Constructive Mathematics

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Year 4 Project
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September 20, 2017

Abstract

We start by exploring the basic principles that govern the logic of constructive mathematics, focusing on the Law of Excluded Middle. We then apply this logic to the real line, and prove a few results that contrast some of the basic theorems of real analysis. Eventually, we prove that every constructive function is not discontinuous.

Thereafter, we focus on choice sequences and non-empty but uninhabited sets. We use the machinery developed here in order to provide a new proof to the Surprise Exam Paradox.

Declaration

I declare that this project was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

Acknowledgement

Many thanks to Dr. Antony Maciocia for all his guidance throughout the course of this project.

Contents

Introduction

At its core, constructive mathematics addresses the notion of existence. As such, it questions the very basic foundations of classical logic from which is derived the bulk of current mathematical knowledge. It is not surprising then that many of the principles and axioms that we take for granted in mathematics are either met with skepticism, or discarded completely in constructive mathematics.

A mathematical object exists to a constructivist only if it can be explicitly constructed, hence the name of the subject. Therefore, a constructive proof of a mathematical statement will almost always involve some type of method or algorithm of generating the desired answer.

It is fair to say that most of the world's mathematical results before the early 19$^{\text{th}}$ century were proven constructively (see [1], pg. 418), albeit without the mathematics community pondering over questions of the validity or invalidity of such proofs. It was David Hilbert, however, who became the first to pioneer the tools and techniques of non-constructive existential proofs. One famous example is his "Hilbert Basis Theorem", which he proved in 1890 using highly non-constructive methods. The German mathematician Hermann Weyl, who was one of Hilbert's students and admirers, defends his teacher by saying that he managed to "inform the world that a treasure exists without disclosing its location" ([17], pg. 6).

Nevertheless, Hilbert would not remain unchallenged. The birth of constructive mathematics came shortly thereafter when "the father of constructive mathematics", L.E.J. Brouwer, rejected the non-constructive logic of Hilbert's proofs. He began laying the foundations for constructive mathematics in his 1907 PhD thesis "On the Foundations of Mathematics", publishing a year thereafter his paper "The Untrustworthiness of the Principles of Logic", in which he questions the validity of the Law of Excluded Middle ([17], pg. 8).

The Law of Excluded Middle states that $P \vee \neg P$, which also validates the method of proof by contradiction. However, in Brouwer's mind, proving the impossibility of an object's non-existence did not guarantee its existence, because there would have been no method presented which would have manifested such an object. Brouwer also reasoned that accepting the Law of Excluded Middle would constructively imply that there could not exist any unsolved mathematical problems, and hence there were "good" reasons for rejecting it (see section 1.3).

Brouwer's new mathematics became known as "intuitionistic mathematics", and the logic that he developed became "intuitionism". The reason for such nomenclature was the belief that constructive mathematics did better than its classical counterpart in encapsulating the "intuition" that the human mind possesses; that which allows us to decide what is mathematically true ([16], pg. 3).

This eventually culminated in the logical formalisation, accepted widely today by most constructivists, known as the "BHK logic", in tribute to its founders; Brouwer, Heyting and Kolmogorov (see section 1.2). It replaced most of classical propositional logic; the essential difference being the meaning behind the existential quantifier. A proof would mean a construction, or method of exhibiting an object in question. An implication would mean a method which transformed a proof of a statement into a proof of the implied statement.

Today, constructivism has become a bliss to the field of mathematical computation and algorithms. All constructive proofs involve computing the existence of an object, and hence form some sort of algorithm. It is realistically applicable, unlike the methods of Hilbert's non-constructivism (see [16]).

Notwithstanding the advantage of applicability, there are a few substantial setbacks when it comes to constructive mathematics. Most classical mathematicians would just simply refuse to accept constructive logic, most probably because it would hinder them from using the Law of Excluded Middle in their proofs. Also, to their dismay, many of their well-established theorems turned out to be constructively false, such as the Extreme Value Theorem and the Intermediate Value Theorem (see section 2.3).

This project contains four main sections. The aim of Chapter 1 is to introduce the basic logic of constructive mathematics to the reader. This is done in order to appreciate the subtle differences that set this subject apart from its classical counterpart.

Chapter 2 aims at using constructive logic on the real line. We will define real numbers and functions from a constructive viewpoint, and use this to prove results that contrast some basic theorems from classical real analysis. More importantly, we will provide an original proof that every constructive function is not discontinuous.

Chapter 3 builds up the machinery of constructive set theory and the notion of choice sequences. We exhibit a trichotomy which captures how "empty" a set is, focusing on the non-empty but uninhabited sets.

Chapter 4 is the culmination of the project, in which we use the tools of Chapter 3 to provide a new and shorter proof, than that in other literature, to the Surprise Exam Paradox. We show that there is indeed no paradox, contrary to what is shown in classical mathematics.

Note to the Reader

1. This project is written under the assumption that the reader has had sufficient exposure to most of undergraduate mathematics.

2. Throughout this project, terminology such as *classical logic* and *classical mathematics* will be used. This refers to the standard logical system that most mathematicians use, in which the Law of Excluded Middle holds, and to the mathematics derived from this system, respectively. When we say that a statement is *constructively false*, we will simply mean that it is constructively not true.

3. Since this is a project in constructive mathematics, we will assume the BHK logic system, unless we are explicitly discussing a statement or proof from classical mathematics. In particular, all our proofs will not assume the Law of Excluded Middle, unless stated otherwise.

4. The lemmas, propositions, and theorems appearing in the text followed by the symbol (*) will denote those statements for which the proof was our original work. This includes, but is not limited to:

    - A statement that we discovered on our own and which has not been found in any of the literature consulted.

    - A known statement but for which the proof in the literature is very different than the proof we will provide.

    - A known statement for which we have not managed to acquire a proof which is appropriate for this project from the consulted literature.

Chapter 1

Logic of Constructive Mathematics

1.1 Basic Logic Symbols of Classical Mathematics

The following table will recall the classical interpretation of logical connectives. We do not include all known connectives; only those relevant for this project. In what follows, A, B, P, $\top$ and $\bot$ will denote statements, and S a set.

Table 1.1: Logic Symbols of Classical Mathematics [18].

Symbol	Name	Meaning
$A \vee B$	Disjunction	At least one of A or B
$A \wedge B$	Conjunction	Both A and B
$\neg P$	Negation	Not P
$A \implies B$	Implication	If A then B
$A \iff B$	Equivalence	If A then B and if B then A
$\exists$	Existential quantifier	$\exists x \in S : P(x)$, there is an x in S for which $P(x)$
$\forall$	Universal quantifier	$\forall x \in S : P(x)$, for every x in S for which $P(x)$
$\top$	Tautology	$\top$ is unconditionally true
$\bot$	Contradiction	$\bot$ is unconditionally false

1.2 Brouwer - Heyting - Kolmogorov (BHK) Logic Symbols

The BHK interpretation is widely accepted by constructivists as the proper version of logical connectives for constructive mathematics. The emphasis is that a statement A is true if it can be explicitly constructed. Consequently, we say that we have a *proof* for A if we have managed to outline an effective method which manifests A. In such a setting, A is called *provable* ([17], pg. 4).

Table 1.2: BHK Logic Symbols ([17], pg. 4)

Symbol	Name	Meaning
$A \vee B$	Disjunction	We have either a proof of A or a proof of B (or both)
$A \wedge B$	Conjunction	We have a proof for both A and B
$A \implies B$	Implication	We have a method that converts any proof of A into a proof of B
$A \iff B$	Equivalence	We have a method that converts any proof of A into a proof of B and vice-versa
$\neg P$	Negation	We have a proof that $P \implies \bot$
$\exists$	Existential quantifier	$\exists x \in S : P(x)$, there is a method that constructs x in S and confirms $P(x)$
$\forall$	Universal quantifier	$\forall x \in S : P(x)$, for every x in S there is a method that confirms $P(x)$
$\top$	Tautology	$\top$ is unconditionally true
$\bot$	Contradiction	$\bot$ is unconditionally false

Remark 1.2.1. *It might appear that both classical and constructive mathematics share the same understanding for what defines a tautology and what defines a contradiction. However, the interpretations of the words* truth *and* false *are different. Constructively, a statement is true if it can be manifested, and false otherwise.*

Remark 1.2.2. *The Law of Excluded Middle (see section* 1.3*) is a classical principle implying that $\neg\bot \iff \top$ and $\neg\top \iff \bot$. This is constructively false as the Law of Excluded Middle is rejected in constructive mathematics (see* [20]*).*

As can be seen in Table 1.2, the main difference between the BHK system and its classical counterpart is what we mean by the existence of a mathematical object. Classically, a non-constructive proof may inform us that a solution to a problem exists, without giving us the solution itself. However, this type of argument is not permitted constructively. Here's an example which illuminates this difference:

Example 1.2.3. (From [12]) *Consider the following quadratic equation in x: $c^2x^2 - (c^2 + c)x + c = 0$, where $c \in \mathbb{R}$.*

A non-constructive proof that this equation has a real solution is to say: if $c = 0$, then any $x \in \mathbb{R}$ is a solution. If $c \neq 0$, then by inspection, $x = \frac{1}{c}$ is a solution. This method is non-constructive because it does not present an explicit solution; it depends on whether or not $c = 0$, which we cannot determine.

A constructive proof, on the other hand, states: $x = 1$ is a real solution. This is an explicit solution independent of c.

Given the BHK framework, one may wonder whether or not the classical results that if $A \implies B$ then $\neg B \implies \neg A$, and vice-versa, still hold. In general, they do not. We present the following proposition without proof. The proof can be found in [19].

Proposition 1.2.4. *Let A and B be mathematical statements.*
If $A \implies B$ then $\neg B \implies \neg A$.
If $\neg B \implies \neg A$, then $A \implies \neg\neg B$.

Remark 1.2.5. *Note that it is constructively false that $\neg\neg B \implies B$. This is because our initial assumption was that we have no proof of B, and the negation* *of this does not mean that we have a proof of B. Not not having a proof does not present us with a method of a proof! This subtlety is the topic of the upcoming section.*

1.3 The Law of Excluded Middle (LEM)

The law of excluded middle is a principle of classical logic used extensively in classical mathematics which states that a statement P is either classically true or false, with no "middle ground", hence the name. It is upon this law that the logic which validates the method of *reductio ad absurdum* (proof by contradiction) is based ([17], pg. 3).

LEM ([17], pg. 3) *Classically this states that for any statement P: $P \vee \neg P$.*

Example 1.3.1. *We use LEM to classically prove the following:*

$$\exists a, b \in \mathbb{R} \backslash \mathbb{Q} : a^b \in \mathbb{Q}$$

Proof. Choose $a = b = \sqrt{2} \in \mathbb{R}\backslash\mathbb{Q}$. Either $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$ or $\sqrt{2}^{\sqrt{2}} \in \mathbb{R} \setminus \mathbb{Q}$, by LEM. If $\sqrt{2}^{\sqrt{2}} \in \mathbb{Q}$, we are done. If not, let $\sqrt{2}^{\sqrt{2}} \in \mathbb{R}\backslash\mathbb{Q}$ be our new a and choose $b = \sqrt{2}$. Now $a^b = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^2 = 2 \in \mathbb{Q}$. In either case, we have managed to find $a, b \in \mathbb{R}\backslash\mathbb{Q}$ that do the job. □

Here is the constructive version of the above proof, from [15].

Proof. It is not too difficult to check, constructively, that $\sqrt{2}$ and $\log_2 9$ are irrational. Let $a = \sqrt{2}$ and $b = \log_2 9$. Then

$$a^b = (2^{\frac{1}{2}})^{\log_2 9} = 2^{\log_2 3} = 3 \in \mathbb{Q}$$

□

Remark 1.3.2. *The constructive proof has a fundamental advantage over its non-constructive counterpart. Namely, it yields explicit results which could be used in other work if necessary. For example, imagine that for some reason we required an actual term $a^b \in \mathbb{Q}$ where a and b are irrational, for some mathematical computation. Then the non-constructive proof would not be of any use since it does not exhibit such a and b, whilst the constructive one does.*

Here is another example illustrating some of the problems that constructivists have with LEM.

Example 1.3.3. (Collatz Conjecture, from [17] pg. 3) *Define a function*

$$f : \mathbb{N} \to \mathbb{N}$$

$$n \mapsto \begin{cases} \frac{n}{2} & \textit{if } n \textit{ is even} \\ 3n+1 & \textit{if } n \textit{ is odd} \end{cases}$$

Then, given any $N \in \mathbb{N}$, $\exists k \in \mathbb{N}$ *such that* $f^k(N) = 1$.

As of today, this conjecture remains unproven. Classically, LEM asserts that the conjecture is either true or false. I.e. $\forall N \in \mathbb{N}$ one will either always find a $k \in \mathbb{N}$ such that $f^k(N) = 1$, or there is some $N_0 \in \mathbb{N}$ for which there is no $k \in \mathbb{N}$ that does the job. And therefore, accepting LEM implies (constructively) that we either always have a method of proving the truth of a certain statement, or we have a method which proves that the statement is false. The point is that we then must possess this method already! Hence we would have solved the problem.

In his 1908 paper "De onbetrouwbaarheid der logische principes" ("The Untrustworthiness of the Principles of Logic"), Brouwer asserts that LEM is constructively valid only if there are no unsolvable mathematical problems. We now give a proof of this assertion, based on ([17] pg. 8).

Metatheorem 1.3.4. (Brouwer, 1908) *LEM holds for every mathematical problem* $P \implies$ *there exists a solution to* P.

Proof. Let P be any mathematical problem. LEM states that $P \vee \neg P$. Assuming LEM, then we either have a method of constructing the truth of P, or we have a proof that $P \implies \bot$. In either case, we must have explicitly found the solution to P. □

In light of his argument, Brouwer dismisses the validity of LEM. However, one must be careful. Constructive mathematics does not claim that for every statement, LEM is false. There may be statements in which LEM is provable. We call such statements *decidable* ([17] pg. 4). Consider the following example:

Example 1.3.5. *Let* $(x_n)_{n=1}^{10}$ *be a sequence of ten binary numbers. Consider the statement P: "either* $x_n = 0$ $\forall n \in \{1, 2, \ldots 10\}$, *or* $\exists N \in \{1, 2, \ldots 10\}$ *such that* $x_N = 1$*".*

Statement P is decidable. Pick some x_i, for $i = 1, 2, 3, \ldots 10$. Then it is either 0 or 1. And since the sequence is finite, either all the x_i's are 0 or there is an $N \in \{1, 2, \ldots, 10\}$ for which $x_N = 1$. Hence for this particular P, LEM is valid, and so P is decidable.

The conclusion to draw here is that even though there may be statements in which LEM is provable, there is no *general* method which proves LEM for every statement S, and therefore it is not included in the BHK interpretation of logic.

Chapter 2

Constructive Mathematics on the Real Line

In this section we highlight the differences between classical and constructive mathematics when it comes to real numbers and functions on the real line.

2.1 The Real Numbers

What is a real number? In classical real analysis, a real number can be defined as a value on the "number line". More informatively, the real numbers are the limits of sequences of rational numbers. Take the real number π as an example. As it stands, π is an infinite object because it has an infinite decimal representation. However, imagine that we wished to compute π up to its 20^{th} decimal point. There is an explicit geometric method of doing so, due to Archimedes. Due to the existence of a method, π is also constructively well-defined. Such an example helps to justify the following definition:

Definition 2.1.1. ([1], pg. 4) *A real number $r \in \mathbb{R}$ admits a sequence of rational numbers (r_n), where $n \in \mathbb{N}$, such that for every $n, m \in \mathbb{N}$,*

$$|r - r_n| \ < \frac{1}{n}, \ |r_m - r_n| \ \leq \frac{1}{m} + \frac{1}{n}$$

We call (r_n) a rational approximation *to r.*

We say $x < y \Leftrightarrow$ for some $n \in \mathbb{N}$ we have

$$x_n + \frac{1}{n} < y_n - \frac{1}{n},$$

where (x_n) and (y_n) are rational approximations to x and y, respectively.

We say $x \leq y \Leftrightarrow y < x$ is not the case.

Remark 2.1.2. 1. *Even though the rational approximations are sequences with infinitely many terms, and constructively we cannot determine an infinity of numbers, the above definition remains constructively well-defined. The reason for this is that it just asks that for every $n \in \mathbb{N}$, the difference between our real number and the n^{th} term of its approximating sequence be at most $\frac{1}{n}$. This is a point-by-point definition and hence does not concern itself with issues of limits as $n \to \infty$.*

2. *A great benefit of such a definition in computer science is its computational applicability. The definition is almost algorithmic, and hence easily programmable.*

However, upon closer inspection, it may seem that there is a major flaw with this definition; it does not address the issue of uniqueness. Certainly, there are many sequences of rational numbers which correspond to the same real number. Constructivists are not too worried about such a thing. Even though we may have two different sequences, as long as the k^{th} term of our first sequence is "close enough" to the k^{th} term of our second sequence, we regard them as "equal".

Definition 2.1.3. ([1], pg. 4) *For real numbers x and y, we say*

$$x = y \iff |x_n - y_n| \ \leq \frac{2}{n} \ \ \forall n \in \mathbb{N},$$

where (x_n) and (y_n) are rational approximations to x and y, respectively.

I will use this definition together with the following lemma to prove another lemma that will become very useful when talking about continuity of real-valued functions.

Lemma 2.1.4. ([4], pg. 37) *For real numbers x and y, $|x + y| \leq |x| + |y|$.*

Lemma 2.1.5. * *For real numbers x and y,*

$$x = y \iff \forall \delta > 0, \ |x - y| \ < \delta$$

Proof. (For a different approach, see [3], pg. 13) If $x = y$ then $|x - y| \ = 0 < \delta$, because $\delta > 0$. If $\forall \delta > 0, \ |x - y| \ < \delta$, then consider rational approximation sequences (x_n) and (y_n) to x and y, respectively. Therefore, there is a choice of subsequences $(\tilde{x}_k) = (x_{n_k})$ and $(\tilde{y}_j) = (y_{n_j})$, defined by picking only the even terms of each sequence, such that

$$|x - \tilde{x}_k| < \frac{1}{2k}, \ \ |y - \tilde{y}_j| < \frac{1}{2j}$$

Note that Definition 2.1.1 implies that $(\tilde{x}_k)$ and $(\tilde{y}_j)$ are also rational approximations to x and y. Working with them, we obtain that, for every $m \in \mathbb{N}$,

$$
\begin{aligned}
|\tilde{x}_m - \tilde{y}_m| \ &= |\tilde{x}_m - x + y - \tilde{y}_m - y + x| && (2.1)\\
&\le |x - \tilde{x}_m| \ + |y - \tilde{y}_m| + |x - y| && (2.2)\\
&< \frac{1}{2m} + \frac{1}{2m} + \delta && (2.3)\\
&= \frac{1}{m} + \delta && (2.4)\\
& && (2.5)
\end{aligned}
$$

where (2.2) was by Lemma 2.1.4. Now we show that, for every $m \in \mathbb{N}$, $|\tilde{x}_m - \tilde{y}_m| \ > \frac{2}{m}$ leads to a contradiction. Indeed,

$$\frac{2}{m} < |\tilde{x}_m - \tilde{y}_m| \ < \frac{1}{m} + \delta$$

for all $\delta > 0$; in particular for $\delta = \frac{1}{m}$. This is a contradiction and so $|\tilde{x}_m - \tilde{y}_m| \ > \frac{2}{m}$ is not the case. Hence, by Definition 2.1.1, $|\tilde{x}_m - \tilde{y}_m| \ \le \frac{2}{m}$ for each $m \in \mathbb{N}$, and so by Definition 2.1.3, $x = y$. □

2.2 The “Axiom” of Trichotomy

Classically, this states:

Axiom 2.2.1. ([7], pg. 6) *Let $x, y \in \mathbb{R}$. Then one and only one of the following is true:*

$$x < y, \ \ x = y, \ \ x > y$$

Constructively however, there are some cases where one cannot be sure what inequality holds for two real numbers x and y.

Example 2.2.2. ([17], pg. 10) *Define a real number $x \in \mathbb{R}$ by*

$$x := \sum_{n \in \mathbb{N}} (-\frac{1}{2})^n a_n$$

where a_n is defined by

$$a_n = \begin{cases} 0 & \text{if 100 zeroes have not appeared by the } n\text{th decimal point of } \pi \\ 1 & \text{otherwise} \end{cases}$$

If we were to constructively claim that $x = 0$ then we would have had to *know* that there are no hundred consecutive zeroes in the decimal expansion of π. If we were to claim that $x > 0$ then we would have known that there are a hundred consecutive zeroes, and they end at an even n. To say $x < 0$ would mean that the hundred zeroes end at an odd n. Hence, admitting the law of trichotomy implies that we have a method of computing exactly where the 100 zeroes occur. Since we do not possess such a method, the law of trichotomy is constructively false.

Remark 2.2.3. *One may mistakenly think that this only shows that we do not know whether $x > 0$, $x = 0$ or $x < 0$ at this point in time; but given enough time and mathematical progress, we will find out and so in fact the law holds. I.e. one may think that this problem is purely time-dependent. This turns out to be constructively false due to a very subtle point. Constructivism deals with existence through a given method, and not with any method that will potentially be given. To see this better, consider the set $S = \{(x < 0)$ or $(x = 0)$ or $(x > 0)\}$. The time argument may suggest that S is not empty. And constructively, that turns out to be true as well! However, that does not mean that S contains any constructible elements, because we do not have the desired method. We call S an* uninhabited *set. This will become clearer when we focus explicitly on such sets in Chapter 3.*

We now make the following crucial definition, inspired by ([1], pg. 11):

Definition 2.2.4. *A* small parameter *t is a real number for which we cannot constructively determine whether $t < 0$, $t = 0$ or $t > 0$; nor whether the negations of these statements hold.*

Notice that in Example 2.2.2 above, x admits the criterion for being a small parameter, and hence small parameters do exist in constructive mathematics. The concept of introducing small parameters becomes very useful when considering real-valued functions on $\mathbb{R}$, especially for providing counter-arguments to the Extreme Value Theorem and the Intermediate Value Theorem. In order to have the right tools to do this, we require the following lemma.

Lemma 2.2.5. * *If t is a small parameter, then given any $\delta > 0$, we have that $|t| \geq \delta$ is not the case.*

Proof. Let t be a small parameter. If it was so that given any $\delta > 0$ we had that $|t| \geq \delta$, then Definition 2.1.1 would imply that $|t| < \delta$ is not the case, and so $t \neq 0$. This is a contradiction because t is a small parameter. Hence $|t| \geq \delta$ is not the case. □

2.3 Functions on $\mathbb{R}$

Classically, a function $f : \mathbb{R} \to \mathbb{R}$ is a mapping that assigns to every value $x \in \mathbb{R}$ a unique value $y \in \mathbb{R}$. We can already see that from a constructive viewpoint, this definition will need some tweaking. Our new definition must take into account the uniqueness of the reals that we established in Definition 2.1.3 ([1], pg. 4). The constructive definition of a function (based on [1], pg. 40) is as follows:

Definition 2.3.1. *Let $I \subseteq \mathbb{R}$. A* constructive function $f : I \to \mathbb{R}$ *is a rule which, for every $x \in I$, determines a real number $f(x) \in \mathbb{R}$ such that, if $x = y \in I$, then $f(x) = f(y) \in \mathbb{R}$.*

Remark 2.3.2.

1. *The definition is similar in spirit to the classical definition, however here we consider f as an explicit method or algorithm which yields real numbers from real numbers, where these numbers must satisfy Definition 2.1.1.*
2. *Note that if $I = \emptyset$, then Definition 2.3.1 is vacuously satisfied. Hence the* empty function*, defined by having $\emptyset$ as its domain, is indeed a constructive function.*

Given Definition 2.3.1, it is a simple task to prove the following lemma:

Lemma 2.3.3. *If $f : \mathbb{R} \to \mathbb{R}$ is a constructive function, then $f|_I : I \to \mathbb{R}$ is also constructive for $I \subseteq \mathbb{R}$.*

We omit the proof as it is very straightforward. Consider the following classically defined step function:

$$S : [0, 2] \to \mathbb{R}$$

$$S(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x \leq 2 \end{cases}$$

Note that S is not a constructive function. The problem lies at the point of discontinuity $x = 1$. The essence of the formal proof goes as follows ([1], pg. 9): consider a small parameter $\delta \in \mathbb{R}$ and consider the real number $1 + \delta$. If f was a constructive function it would tell us whether $f(1 + \delta) = 0$ or $f(1 + \delta) = 1$, in which case we would know whether $\delta < 0$ or $\delta \geq 0$. This contradicts the fact that δ is a small parameter. Hence f is not a constructive function.

This example highlights a vital question: is it possible to have a constructive function that is discontinuous? After all, the above idea appears to be applicable to every discontinuous function: pick the point of discontinuity, add the small parameter and arrive at some sort of a contradiction. Also, is it true that every constructive function must be continuous? Brouwer strongly believed this. He devoted lots of his time to justify this theorem, and many intuitionists accepted his proof of it. However, a notable constructivist, Errett Bishop, did not accept Brouwer's proof but did not manage to prove the theorem himself, choosing to bypass it entirely ([1], pg. 10).

Inspired by this discussion, we proceed to give a negative answer to our first question. In order to do so, we require a few definitions.

Definition 2.3.4. *Let $I \subseteq \mathbb{R}$. Let $f : I \to \mathbb{R}$ be a constructive function. We will say that f is* discontinuous *if:*

1. *There is a point $x_0 \in I$ such that*

$$\exists\epsilon > 0 \text{ such that } \forall\delta > 0,\ \exists x \in I \text{ such that } |x-x_0| < \delta \text{ but } |f(x)-f(x_0)| > \epsilon$$

2. *If for any $y \in I$ and for all $\delta > 0$ the claim that:*

$$y \notin \{x \in I\ :\ |x| < \delta \text{ but } |f(x)| > \epsilon\}$$

is false (where $\exists\epsilon > 0$), then we have that $\exists\delta > 0$ for which

$$y \in \{x \in I\ :\ |x| < \delta \text{ but } |f(x)| > \epsilon\}$$

Remark 2.3.5. *1. Note that if $I = \emptyset$, then to say $\exists x \in \emptyset$, as in point 1 of the definition, is a false statement. Hence the empty function is not discontinuous.*

2. The second point is classically always true, but not so constructively as $\neg\forall$ does not generally imply $\exists$. However we assume that the discontinuous constructive functions are the ones with this additional property, in order to ease the argument in Theorem 2.3.8.

Definition 2.3.6. *Let $I \subseteq \mathbb{R}$. We call a function $f : I \to \mathbb{R}$ that is not discontinuous a* weakly continuous *function.*

Remark 2.3.7. *One might falsely think that a function which is not discontinuous must be continuous. However, recall that Remark 1.2.5 stated that $\neg\neg P$ does not imply P.*

Theorem 2.3.8. * *Every constructive function $f : \mathbb{R} \to \mathbb{R}$ is weakly continuous.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a constructive function. We assume that f is discontinuous, and show that this leads to a contradiction, whence f must be weakly continuous. If f is discontinuous, we can, without loss of generality, make vertical and horizontal shifts to f such that one of its points of discontinuity lie on $x = 0$, and such that $f(0) = 0$. Now we can state, thanks to Definition 2.3.4:

$$\exists\epsilon > 0 \text{ such that } \forall\delta > 0,\ \exists x \in \mathbb{R} \text{ such that } |x| < \delta \text{ but } |f(x)| > \epsilon$$

Fix a $\delta > 0$. Set $I_\delta = \{x \in \mathbb{R},\ \text{such that } |x| < \delta \text{ but } |f(x)| > \epsilon\}$. Note that for every $\delta > 0$, $\exists x \in I_\delta$, since f is assumed to be discontinuous, and so $\forall\delta > 0$, $I_\delta \neq \emptyset$.

Let t be a small parameter. I will show that ($\forall\delta > 0$ we have that $t \notin I_\delta$) is a contradiction. So assume that $\forall\delta > 0$, $t \notin I_\delta$. Then this implies one of two cases. Either $\forall\delta > 0$ we have that $|t| < \delta$ but $|f(t)| \leq \epsilon$, or that $\forall\delta > 0$ we have that $|t| \geq \delta$. The latter is not the case however, due to Lemma 2.2.5. Hence it can only be the first case that holds. But that holds if and only if $t = 0$, by Lemma 2.1.5 with x replaced by t and y by 0. However, we cannot claim that $t = 0$, as t is a small parameter. Thus we have a contradiction and so Definition 2.3.4 implies that $\exists \delta > 0$ such that $t \in I_\delta$.

Therefore, it must be that $f(t) > \epsilon$. This directly implies that $t \neq 0$, which in turn contradicts the fact that t is a small parameter. Thus, f is not discontinuous and hence weakly continuous.

□

Remark 2.3.9. 1. *Even though the above proof was not found in any of the consulted literature, the actual theorem is widely known and accepted constructively (see* [2]*, pg.157).*

2. *In the field of computational mathematics, it is in fact similarly so that there are no discontinuous* computable *functions. We do not define computable functions here, but for more information and a statement of the theorem, see* [9]*.*

2.3.1 The Extreme Value Theorem

Classically, this states that every continuous function on a bounded interval of $\mathbb{R}$ achieves its maximum and minimum on that interval. Constructively, this is false.

Theorem 2.3.10. ([1], pg. 10) *The classical Extreme Value Theorem is constructively false.*

Proof. Consider a constructive function $f : \mathbb{R} \to \mathbb{R}$ with two relative maxima, one at $x_1 = 0$ and another at $x_2 = 2$ (See Figure 2.1). We can, without loss of generality, work with maxima rather than minima. Assume, without loss of generality, that $f(0) = 1$ and $f(2) = 1 + t$ for a small parameter t. If we gave a proof that the maximum is either at x_1, x_2, or both, then we would have constructively determined whether $t > 0$, $t = 0$, or $t < 0$; which is impossible as t is a small parameter. Hence the Extreme Value Theorem is constructively false. □

Remark 2.3.11. *One may think that such a function is not constructive because declaring that $f(2) = 1 + t$, for t a small parameter, cannot be a rule since we do not know whether $t < 0$, $t = 0$ or $t > 0$. However, all Definition 2.3.1 requires is a method which yields a real number from a real number, where we are implicitly using Definition 2.1.1 in interpreting what it means for a number to be a real number. Clearly,* 2 *is real number, and so is $1 + t$ because t is real number by Definition 2.2.4. So in fact, f is constructive.*

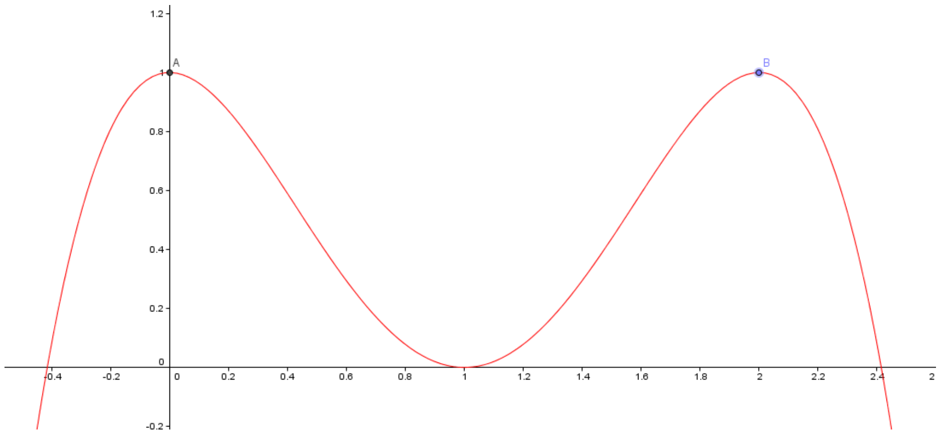

Figure 2.1: A pictorial example of a constructive function which fails the Extreme Value Theorem.

2.3.2 The Intermediate Value Theorem

Classically, this states that if a continous function $f : I \to \mathbb{R}$, where $I = [a, b] \subset \mathbb{R}$ is a bounded interval, satisfies $f(a) < 0$ and $f(b) > 0$ then $\exists c \in [a, b]$ such that $f(c) = 0$. It turns out that this is constructively false.

Theorem 2.3.12. ([1], pg. 11) *The classical Intermediate Value Theorem is constructively false.*

Proof. Without loss of generality, we will work with the bounded interval $[0, 3]$. Consider a constructive function $f : [0, 3] \to \mathbb{R}$ which takes the value -1 at $x = 0$ and 1 at $x = 3$ and is piecewise linear, taking the constant value t for a small parameter t between $x = 1$ and $x = 2$ (See figure 2.2). Note that if the Intermediate Value theorem holds, then we would be able to tell whether $t < 0$, $t = 0$ or $t > 0$. But t is a small parameter, and hence the Intermediate Value Theorem does not always hold on constructive functions.

□

Remark 2.3.13. *There seems to be a link between Constructive Mathematics/Classical Mathematics and Quantum Mechanics/General Relativity in the field of Physics. Indeed, we have seen that classical mathematics behaves more "idealistically" compared to its constructive counterpart. The Law of Excluded Middle plays a key role in such a phenomenon (see section 1.3). Things are nicely packaged into "true" or "false" in classical mathematics, whilst, as we have seen in this section, things are much more subtle in constructive mathematics. Likewise, General Relativity behaves very deterministically in Physics, whilst some of its very basic principles start to falter at the quantum level, much in the same way that some of the basic principles or axioms of classical mathematics stop making sense at the constructive level. (For a more elaborate discussion of this, see* [13]*). Perhaps classical mathematics is better fit to explain the general structure of nature's rules, whilst constructive mathematics helps us understand the details.*

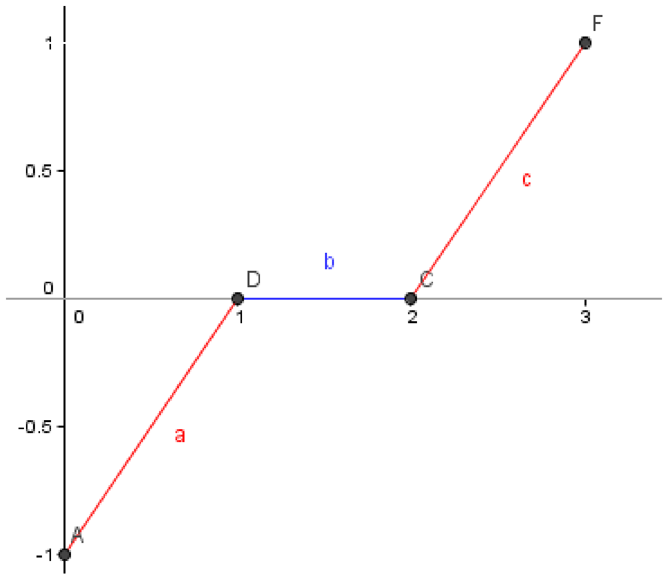

Figure 2.2: A pictorial example of a constructive function which fails the Intermediate Value Theorem.

Chapter 3

Choice Sequences and Sets in Constructive Mathematics

3.1 Limited Principle of Omniscience

Recall Example 1.3.5 above. It was quite straightforward to prove that either the ten binary numbers are all 0, or there is at least one which equals 1. It is not hard to believe that in fact a similar method works for any binary sequence of finite size. What if we had an infinite sequence of binary numbers? The method we described in Example 1.3.5 would fail because constructively, we cannot consider an infinity of numbers. The sequence may be of the form $0, 0, 0, 0, \ldots$ where we cannot claim that *all* the terms of the sequence are 0, nor that there will appear a 1. The reason for this is that there does not exist a method that is capable of searching through an infinitude of numbers. In light of this example, we state the classical principle of omniscience:

Principle of Omniscience (PO) ([10], pg. 1) *Let X be a set. For every function*

$$f : X \to \{0, 1\}$$

we have,

$$(\exists x \in X \textit{ such that } f(x) = 0) \ \vee \ (\forall x \in X \textit{ we have that } f(x) = 1)$$

We state now the Limited Principle of Omniscience based on the above:

Limited Principle of Omniscience (LPO) *For every function*

$$f : \mathbb{N} \to \{0, 1\}$$

we have that,

$$(\exists \text{ a minimal } n \in \mathbb{N} : f(n) = 0) \vee (\forall n \in \mathbb{N} \text{ we have that } f(n) = 1)$$

We will prove that the LPO fails constructively, but in order to do so, we need to equip ourselves with some knowledge of free choice sequences.

3.2 Free Choice Sequences

Intuitively speaking, a *choice sequence* is an infinite sequence of natural numbers, one number constructed after the other. The word *choice* refers to the idea that every subsequent number is generated by a choice of some sorts. The choice may be influenced by some effective law or rule, or it may not, in which case our choice sequence becomes a *free choice sequence.* We now define a choice sequence:

Definition 3.2.1. ([11], pg. 91) *A* choice sequence *is an infinite sequence of natural numbers, whose terms are generated in succession. A* choice sequence based on $S \subset \mathbb{N}$ *is an infinite sequence of elements from* S*, whose terms are generated in succession.*

Definition 3.2.2. ([11], pg. 91) *A* lawlike choice sequence *(or a* lawlike choice sequence based on $S \subset \mathbb{N}$*) is a choice sequence (or a choice sequence based on* $S \subset \mathbb{N}$*) where there exists an effective rule* $P : \mathbb{N} \to \mathbb{N}$ *(or an effective rule* $P : \mathbb{N} \to S$*) such that* $P(n)$ *is the* n^{th} *term of the sequence. A* lawless choice sequence *(or a* lawless choice sequence based on $S \subset \mathbb{N}$*) is a choice sequence (or a choice sequence based on* $S \subset \mathbb{N}$*) that is not lawlike.*

We use these definitions to define a free choice sequence.

Definition 3.2.3. *A* free choice sequence *is a lawless choice sequence. A* free choice sequence based on $S \subset \mathbb{N}$ *is a lawless choice sequence based on* $S \subset \mathbb{N}$*.*

For a slightly different, yet consequently equivalent definition see [6] (pg. 2). Brouwer was the first to introduce the idea of a free choice sequence. He named the constructing agent (who is free at any stage of the sequential progression to choose the next term) a *creative subject* ([17], pg. 8). We illustrate these ideas with an example.

Example 3.2.4. *Let* $S = \{0, 1\} \subset \mathbb{N}$*. Here is a lawlike choice sequence based on* S*:*

$$0, 1, 0, 1, 0, 1, 0, 1, 0, 1 \ldots$$

subject to the law

$$n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

where n is the n^{th} term in the squence, for $n = 1, 2, 3, \ldots$

Here are the first 10 terms of a free choice sequence based on S:

$$0, 0, 0, 1, 0, 1, 1, 1, 0, 0, 1, \ldots$$

Proposition 3.2.5. *Let $S = \{0, 1\} \subset \mathbb{N}$. Then for all free choice sequences $f : \mathbb{N} \to S$, the limited principle of omniscience (LPO) is constructively false.*

Proof. (Based on the idea in [17], pg. 9) Recall that LPO states that if $f : \mathbb{N} \to \{0, 1\}$, then either ($\exists$ a minimal $n \in \mathbb{N} : f(n) = 0$) or ($\forall n \in \mathbb{N}$ we have that $f(n) = 1$). Let $f : \mathbb{N} \to \{0, 1\}$ be a free choice sequence, and assume that LPO holds. We want to arrive at a contradiction.

Clearly, the statement that ($\forall n \in \mathbb{N}$ we have that $f(n) = 1$) cannot hold, because if it did, then we would have a method of determining $f(n) = 1\ \forall n \in \mathbb{N}$, which means that f is a lawlike choice sequence. Hence it remains to show that the claim ($\exists$ a minimal $n \in \mathbb{N} : f(n) = 0$) is constructively false as well.

Assume that the claim holds. That means there is a method of constructing a smallest $n \in \mathbb{N}$ which proves $f(n) = 0$. Equivalently, this means there exists a constructive function $a : S^{\mathbb{N}} \to \mathbb{N}$, such that if f is a free choice sequence based on S, we have that $a(f) = N$, where $N = \min\{n \in \mathbb{N} : f(n) = 0\}$. From this we could conclude that $f(1) = 1, f(2) = 1, \ldots, f(n-1) = 1, f(n) = 0$. Consider now only the terms of the free choice sequence following the first n terms. Call this new sequence $f_1 = (f(n+1), f(n+2), \ldots)$. Apply a to f_1 and repeat the process of determining where the zeroes lie in the sequence. Combining the results and repeating this process means that there is an effective rule, P, such that $\forall n \in \mathbb{N}$, we have that $P(n) = f(n)$, and hence f cannot have been a free choice sequence. Therefore, neither of the two possibilities of LPO hold and hence LPO is constructively false. □

The concept of free choice sequences also leads to one of the most fundamental principles of constructive mathematics, that of *continuous choice.*

Principle of Continuous Choice (PCC) ([8], pg. 680) If $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$, and for each $s \in \mathbb{N}^{\mathbb{N}}$ there is an $n \in \mathbb{N}$ such that $(s, n) \in P$, then there is a *continuous choice function*

$$c : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$$

such that $(s, c(s)) \in P, \ \forall s \in \mathbb{N}^{\mathbb{N}}$.

Remark 3.2.6. *In some of the literature, the Principle of Continuous Choice includes, in its definition, what is known as* **Cont**$(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$*, which is a principle of Brouwer's that states that every constructive function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is continuous (see* [1]*, pg. 369, and* [14]*, pg. 6). However, we are omitting the inclusion of* **Cont**$(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$ *because we will show in Proposition 3.2.7 that it is in fact just a consequence of PCC.*

Intuitively, PCC can be interpreted as follows. Imagine we were given a sequence s of natural numbers, $s = (x_1, x_2, x_3, \ldots)$. PCC then says that if every such sequence can be paired with some natural number n, then there is a continuous method of pairing every similar sequence with a natural number. Imagine that there was a computer going through the sequence, term by term. It would eventually stop due to lack of infinite memory capacity. Hence, constructively we cannot determine all the terms of s. But given an arbitrary $k \in \mathbb{N}$, we can perhaps determine the first k terms of s. Now say that we were presented with a second infinite sequence y, where the first k terms of y agree with those of s. Then PCC says that we could pair y to the same natural number. Hence the "difference" between s and y is seen to be "sufficiently small" so that $c(s) = c(y)$, and hence the term *continuous* is justifiable.

We now use our definition of PCC to prove $\mathbf{Cont}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$.

Proposition 3.2.7. * *Assuming PCC, every constructive function $f : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is continuous.*

Proof. Let f be as in the statement of the proposition. Define

$$P = \{(a, b) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \mid b = f(a)\}$$

First note that $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}$. Also note that for every $s \in \mathbb{N}^{\mathbb{N}}$ there exists an $n \in \mathbb{N}$, namely $f(s)$, such that $(s, n) \in P$. Hence by PCC, there is a continuous function $c : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ such that $\forall s \in \mathbb{N}^{\mathbb{N}}$, $(s, c(s)) \in P$. This implies that $f(s) = c(s)$ for every $s \in \mathbb{N}^{\mathbb{N}}$ and hence $c = f$. As c is continuous, so is f. $\square$

3.3 Non-empty but uninhabited sets

The Riemann Hypothesis is one of the most famous unsolved mathematical conjectures. To this day mathematicians have not been able to prove it, nor find a counter-example to disprove it. Riemann hypothesised that the non-trivial zeroes of the Riemann ζ-function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\mathrm{Re}(s) > 1$, all have real part equal to $\frac{1}{2}$.

Consider a set S, defined as follows:

$$S = \{(1 \text{ if the Riemann hypothesis is true}) \vee (0 \text{ if the Riemann hypothesis is false})\}$$

What properties does S have? First of all, note that S cannot be empty, because if it was, then

$$(\neg\text{Riemann hypothesis is true}) \wedge (\neg\text{Riemann hypothesis is false})$$

which is a clear contradiction. Hence, S is non-empty. However, in order to exhibit an element of S, one would need to find a method which would explicitly tell us if 1 or 0 was in S, and in doing so, would have proved or disproved the Riemann hypothesis. Hence we cannot exhibit a member of S. S is an *uninhabited* set. We will define, more precisely, the notion of a set being inhabited (or uninhabited) in the upcoming section.

Such sets are of great importance in constructive mathematics. Not only do they highlight the subtlety of the subject, but also once again cast doubt on the validity of LEM. They will also play a crucial role in translating problems, which may be quite difficult to solve otherwise (such as the Surprise Exam Paradox) into the language of a non-empty but uninhabited set. As we shall see, studying these sets will be the first step in providing a new and shorter proof to the paradox.

Let P be a syntactically correct statement and consider the following set:

$$S = \{(1 \in \mathbb{N} \ \wedge \ P) \ \vee \ (0 \in \mathbb{N} \ \wedge \ \neg P)\}$$

If S is empty, then $\neg P \ \wedge \ \neg\neg P$ which is $\perp$. Hence we conclude that $S \neq \emptyset$. But to say $1 \in S$ would be to prove (using the BHK interpretation) P, and to say $0 \in S$ would be to prove $\neg P$. If one was able to do that for all P then this would be a constructive proof of the validity of LEM, which we know is constructively false. Hence there are some P for which we cannot construct any members of S.

Definition 3.3.1. ([17], pg. 7) *A set which has constructible members is called* inhabited*, and a set which is not inhabited is called* uninhabited*.*

Hence the set from the above example is non-empty but uninhabited. In order to shorten writing, we introduce the following definitions:

Definition 3.3.2. *A set which is empty is called a* 1-empty set*. A set which is inhabited is called a* (-1)-empty set*. A set which is non-empty but uninhabited is called a* 0-empty set*.*

Lemma 3.3.3. * *Every set S is one, and only one, of 0, 1, or (-1) -empty.*

Remark 3.3.4. *The reason this trichotomy of emptiness was not found in any of the consulted literature is because this terminology is purely our creation in order to neaten some of the subsequent arguments up.*

Proof. Let S be any set. We have that either S is empty or S is non-empty. If S is empty, then to say that it is inhabited would mean that we have managed to manifest one of its members, which cannot happen. And S being empty means that it is not non-empty. Also, being empty easily implies that S is uninhabited. Hence S is only 1-empty.

If S is non-empty, then it is either inhabited or uninhabited. If it is inhabited, then it cannot also be uninhabited, and hence S would be -1-empty. If it is uninhabited, then S is non-empty but uninhabited which by definition is 0-empty. Since a set S must be either empty or non-empty, and inhabited or uninhabited, this exhausts all options and proves the result. □

We now extend these ideas to products of sets.

Definition 3.3.5. *Let $S_1, S_2, \ldots S_k$ be sets, where $k \in \mathbb{N}$. Then their* product *is defined by*

$$S_1 \times S_2 \times \ldots \times S_k = \{(s_1, s_2, \ldots, s_k) \mid s_i \in S_i \text{ for each } i \in \{1, 2, \ldots, k\}\}$$

When $S_1 = S_2 = \ldots = S_k := S$, we denote this product by S^k.

We will now state, without proof, a lemma extracted from [5]:

Lemma 3.3.6. ([5], pgs. 25, 59) *Let A and B be sets. Then:*

1. *A or B is $\emptyset \iff A \times B = \emptyset$.*
2. *$A \neq \emptyset \neq B \iff A \times B \neq \emptyset$.*

Using the above lemma from Halmos's book, we prove the following proposition concerning the trichotomy of emptiness for products of sets:

Proposition 3.3.7. * *Let S be a set. Then S is j-empty $\iff$ S^k is j-empty for every $k \in \mathbb{N}$, and $j \in \{-1, 0, 1\}$.*

Proof. Let S be a set and $k \in \mathbb{N}$, and let $j \in \{-1, 0, 1\}$. ($\Rightarrow$) Assume S is j-empty. If $j = -1$ then S is inhabited and so we have a method of manifesting the existence of an element $s \in S$. Let $x = (s, s, \ldots, s) \in S^k$ which we have explicitly constructed. Then $x \in S^k$ exists and so S^k is inhabited, hence (-1)-empty. Let $j = 0$. Then even though S is non-empty, we have no method of exhibiting an element of it. To say $x = (s_1, s_2, \ldots, s_k) \in S^k$ is thus false because it would imply by definition that we have constructed $s_j \in S$, which we cannot do. Hence S^k is uninhabited. Furthermore, we have that S^k is non-empty because S is non-empty (Lemma 3.3.6). Hence S^k is 0-empty. Let $j = 1$. Thus $S = \emptyset$ and so $S^k = \emptyset$ by Lemma 3.3.6. Therefore S^k is 1-empty.

($\Leftarrow$) Let S^k be (-1)-empty. Hence $\exists x = (s_1, s_2, \ldots, s_k) \in S^k$ and so we can construct an $s_i \in S$. So S is (-1)-empty. Let S^k be 0-empty. Therefore we cannot exhibit any element of S^k and therefore no element of S. Thus S is uninhabited. As S^k is non-empty, so is S by Lemma 3.3.6. Hence S is 0-empty. Let S^k be 1-empty. Then S is empty by Lemma 3.3.6, i.e. 1-empty. We have exhausted all the cases and thus the result holds. □

We now prove a crucial result that will become the key ingredient in providing the shorter proof to the Surprise Exam Paradox.

Lemma 3.3.8. * *Assume that*

$$S = \{c = (c_1, c_2, \ldots, c_5) \mid c \text{ is a finite free-choice sequence based on } \{0, 1\}\}$$

Let $\emptyset \neq A \subseteq S$. Then A is 0-empty.

Proof. By assumption we have that A is non-empty. Hence A cannot be 1-empty. If we show that (A is (-1)-empty) $\implies \bot$, then by the trichotomy of possibilities proved in Lemma 3.3.3, A must be 0-empty.

Assume A is (-1) -empty. Hence, A contains a constructible member, call it τ. This means that we have a method P which computes $\tau(n) = P(n)$ for $n \in \{1, 2, 3, 4, 5\}$. This means that τ is a lawlike choice sequence. However, we know that τ is a free-choice sequence and hence a lawless choice sequence. This is a contradiction, and thus, A cannot be (-1)-empty. Hence it is 0-empty. □

Chapter 4

A New Proof of the Surprise Exam Paradox

A teacher announces to his students: *There will be one and only one exam next week between Monday and Friday at 10.00AM, but you will not be able to know in advance the day of the exam* (from [8], pg. 679).

A clever student argues that this is false: *If the exam was on Friday, then after Thursday's class is over everyone will know that the exam will happen the next day. If it's not on Friday but on Thursday, then after Wednesday's class is over, everyone would have reasoned that the teacher would not have chosen Friday by the above argument, and hence the remaining option would be Thursday; and thus they would know the day of the exam. By repeating this backward induction argument, the exam must be on Monday and hence the students will in fact be able to determine when the exam is. Therefore the teacher's claim is false.*

The paradox is that it seems absurd that the students will be able to determine when the exam will happen before it happens, and yet the student's argument sounds plausible.

I will sketch a constructive proof, due to the authors of [8], that shows that there is no paradox in the above statement and that the students cannot in fact know the time of the exam. The aim of this is to illustrate how cumbersome this particular constructive proof actually is, thus motivating our search, in the subsequent sections, for a shorter and more elegant proof using the tools we developed in Chapter 3.

4.1 Sketch of a Long Proof

We start by reformulating the teacher's claim using the notion of choice sequences. In order to do that, we need to define what we mean by a *maximum* of a choice sequence.

Definition 4.1.1. ([8], pg. 681) *Let α be a choice sequence. Its* maximum*, $\max(\alpha, k)$, is the natural number $k \in \mathbb{N}$ satisfying:*

$$\forall n \in \mathbb{N},\ (\alpha(n) \leq k) \wedge (\exists n \in \mathbb{N} \text{ such that } \alpha(n) = k).$$

Now we can state the teacher's claim formally:

Teacher: *I will create a choice sequence α based on $\mathbb{N}$ as follows. I will pick an arbitrary $n_1 \in \mathbb{N}$, and I will declare $\alpha(1) = \alpha(2) = \ldots = \alpha(n_1) = 1$. If I decide to have the exam on Monday, I will declare $\alpha(n) = 1\ \forall n > n_1$. Otherwise I will pick an arbitrary $n_2 \in \mathbb{N}$ and declare $\alpha(n_1+1) = \alpha(n_1+2) = \ldots \alpha(n_1+n_2) = 2$. If I decide to have the exam on Tuesday then I will declare $\alpha(n) = 2\ \ \forall n > n_1+n_2$. I reiterate this process, if necessary, until $\alpha(n) = 5\ \ \forall n > n_1+n_2+n_3+n_4+n_5 \in \mathbb{N}$. You (the students) will not be able to determine the maximum of this sequence α.*

What follows is the summary of the long proof provided in [8], that verifies the teacher's claim:

1. A *spread*, T, is defined, and it is shown that every choice sequence that the teacher may produce must belong to T. We do not give the definition of a spread as it is beyond the scope of this project.

2. Assuming (for a contradiction) that the student's reasoning is correct, it is shown that $\forall \alpha \in T$, one obtains that

   $$\neg\neg\exists m \in \mathbb{N} \text{ such that } \forall n \geq m, \text{ where } n \in \mathbb{N}, \text{ we have that } \alpha(n) = \alpha(m).$$

   Note that this simply states that assuming that a maximum does not exists is false.

3. An easy corollary is proved, and it states that for all $\alpha \in T$,

   $$\neg\neg\exists k \in \mathbb{N} \text{ such that } (\max(\alpha, k)).$$

4. Using the principle of continuous choice (PCC), it is proved that there does not exist a function $f : T \to \mathbb{N}$ such that

   $$\forall \alpha \in T\ (\max(\alpha, f(\alpha))).$$

   Hence a maximum cannot be determined and thus there is no paradox.

There are a few setbacks with the above proof. It is very long and is based highly on the notion of *trees* and *spreads*, neither of which we wish to include. Thus the aim is to write an alternative proof that uses only the concepts introduced in Chapter 3. The non-rigorous motivation for this is as follows:

- If we manage to somehow construct a set structure on the paradox, then the teacher claiming that *there will be* an exam is analogous to our potential set being *non-empty*. However the claim that the students will not be able to *determine* when the exam will take place is analogous to this set being *uninhabited.*

- The student's claim that the day of the exam can be predicted is analogous to the set being *inhabited.*

4.2 The New Proof

Consider labeling the days of the week in order as 1,2,3,4, and 5. Rather than considering a maximality argument on a choice sequence as presented in the previous proof, we consider another equivalent mathematical statement to the teacher's claim:

Teacher: *I will construct a finite free-choice sequence c, based on* $\{0,1\}$*, as follows. I will set* $c = (c_1, c_2, \ldots, c_5)$ *, and if I decide to have the exam on day* $k \in \{1,2,3,4,5\}$ *of the week, I will declare* $c_k = 1$ *and I will set the other terms of my sequence to 0. You (the students) will not be able to determine what my sequence is.*

This is indeed an equivalent statement to the teacher's original statement because it is saying that *there will* be an exam (the teacher will declare a 1 somewhere), but he claims that the students will not know when it is (cannot determine the sequence).

Note also that in order to prove the teacher's claim, we will need c to indeed define free-choice sequence based on $\{0,1\}$, as the teacher claims it does. Here is the proof for that:

Lemma 4.2.1. * *The sequence defined by the teacher is a free choice sequence based on* $\{0,1\}$*.*

Proof. Assume that c, as defined by the teacher, was a lawlike choice sequence. That would imply that there is an effective rule, $P : \{1,2,3,4,5\} \to \{0,1\}$, which determines every term in the sequence, $P(k) = c(k)$. And hence $\exists k \in \{1,2,3,4,5\}$ such that $P(k) = 1$, and so the teacher *will* choose the exam to be on day k. But by the definition of the sequence, the teacher was free to declare $P(k) = 0$, and so this is a contradiction. Hence c is not lawlike, which by definition means that it is a free choice sequence. □

In this language, the student's claim is as follows
Student: *There exists a method of determining the sequence created by the teacher.*

We now use Lemma 4.2.1 to provide the new proof to the surprise exam paradox.

Theorem 4.2.2. * *The surprise exam paradox is not a paradox. In particular, the teacher's claim is true, and the student's reasoning is false.*

Proof. Assume that

$$S = \{c = (c_1, c_2, \ldots, c_5) \mid c \text{ is a finite free-choice sequence based on } \{0, 1\}\}$$

Clearly, the set of sequences that the teacher may come up with, call it A, contains only free-choice sequences by Lemma 4.2.1, and so we have that $\emptyset \neq A \subseteq S$. Lemma 3.3.8 now implies that A is 0-empty and thus it has no constructible members.

In particular, the teacher's claim that one will not be able to determine this sequence holds, and the student's claim that one will be able to determine it is false. This completes the proof. □

Remark 4.2.3. *We keep alluding to the fact that our proof is shorter than the one in* [8]*. However, it is so that to get to this stage we had to develop a wide machinery of constructive set and sequence theory. By saying that our proof is short we mean that it is a quick and easy consequence of this machinery.*

Chapter 5

Some Final Remarks

We have clearly seen, throughout the progression of this project, that proving statements constructively is much more subtle than doing it non-constructively, due to the removal of the Law of Excluded Middle. The examples we included in this project highlighted that, for some cases, the non-constructive approach was quicker and more elegant. However, we showed that for some other cases, it is much more fruitful proving things constructively.

We have fulfilled the tasks we set out to achieve in the commencement of the project. We wished to discuss the differences that arise in the logic of classical mathematics compared to that of constructive mathematics, and we have extensively done so. We aimed to focus our analysis on the real line, and highlight the differences between classical real analysis and constructive analysis, and we did so. We set as a goal to prove some continuity results for real-valued functions, and we managed to do so with an original approach. We intended to develop a toolbox of results concerning sets and sequences in constructive mathematics in order to provide a new proof to the Surprise Exam Paradox, and we managed to do just that.

It is in our hopes that mathematicians consider and focus on constructive mathematics in their future research. First of all, the modern technological world is arguably run mainly by computer programmes and hardware. However, the theory relies heavily on mathematics. Constructive proofs provide programmable results, and hence will certainly advance this technological development. Secondly, mathematics is, after all, a man-made subject. We define things based on our intuition, and derive results from such definitions. It is therefore arguably closer to our intuitive nature to approach things with intuitionistic logic. And that is where constructive mathematics starts playing a fundamental role.

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