Almgren's frequency function and unique continuation

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Contents

Introduction and Statement of the Theorems

The aim of this essay is to prove strong unique continuation for weak solutions $u \in W^{1,2}_{\text{loc}}(\Omega)$ to a certain elliptic operator L (Ω and L will be defined later). That is, if u satisfies $Lu = 0$ and vanishes to infinite order at a point in Ω, then u is identically 0 in Ω. We will denote by D_r the open ball in $\mathbb{R}^n$ of radius r, and by B_r the closed ball of radius r, centred at the origin.

Definition 0.0.1. *A function $u \in L^2_{\text{loc}}(\Omega)$ is said to* vanish to infinite order *at $x_0 \in \Omega$ if for all $\delta > 0$ sufficiently small and natural numbers $j \in \mathbb{N}$ we have that*

$$\int_{D_\delta(x_0)} u^2 = \mathcal{O}(\delta^j). \tag{1}$$

The aim is thus to prove

Theorem 0.0.2 (Strong Unique Continuation). *If a solution $u \in W^{1,2}_{\text{loc}}(\Omega)$ to equation (7) defined below vanishes to infinite order at $x_0 \in \Omega$, then u is identically 0 in Ω.*

To achieve this we are going to make use of the doubling condition:

Theorem 0.0.3 (Doubling Condition). *Assume that $u \in W^{1,2}_{\text{loc}}(\Omega)$ solves equation (7). Then there exists a constant C depending on $n, \mathcal{K}, \lambda$ and u such that for every $0 \leq R < \frac{1}{2}$ we have*

$$\int_{B_{2R}} u^2 \leq C \int_{B_R} u^2. \tag{2}$$

The constants λ and $\mathcal{K}$ will be introduced shortly. We will not assume the doubling condition, and so in order to prove it we need to prove that the modified Almgren frequency function is non-decreasing - see Theorem 0.0.5 below.

These results appear in [GL86] and this essay will follow the paper of Garofalo and Lin closely. However, the proofs in [GL86] are very dense (some derivations are stated without proof) and require a lot of prerequisite knowledge of tensor calculus on manifolds, none of which we assume. Hence this essay will prove the same results but will include a lot of what [GL86] leaves out. We also include a section on tensor calculus in Chapter 1. A novelty in this essay is a proof that the Almgren frequency function is non-decreasing for the Laplacian operator (see Theorem 0.0.6 below) without any use of the differential geometry in [GL86]. All that follows in this introduction is based on the setup adopted in [GL86].

- Ω is a smooth, connected and bounded open subset of $\mathbb{R}^n$ for $n \in \mathbb{N}$ with $n \geq 3$. We will henceforth assume that

$$B_2 = \{x \in \mathbb{R}^n \text{ such that } |x| \leq 2\} \subset\subset \Omega. \tag{3}$$

- L is the operator defined by

$$L = \text{div}(A\nabla), \tag{4}$$

  where $A(x) = (a_{ij}(x))$ is a symmetric $n \times n$ matrix on Ω. L satisfies the following conditions:

  1. The entries of A are real and Lipschitz continuous. That is, $\exists \mathcal{K} > 0$ such that $\forall x, y \in \Omega$ we have that

$$|a_{ij}(x) - a_{ij}(y)| \leq \mathcal{K}|x - y| \text{ for all } i, j \in \{1, 2, \dots n\}. \tag{5}$$

  2. L is a strictly elliptic operator. In particular, we assume $\exists \lambda \in (0, 1)$ such that for every $x \in \Omega$ and for all $\zeta \in \mathbb{R}^n$ we have

$$\lambda|\zeta|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\zeta_i\zeta_j \leq \frac{1}{\lambda}|\zeta|^2. \tag{6}$$

- We consider non-trivial weak solutions $u \in W^{1,2}_{\text{loc}}(\Omega)$ to the equation

$$Lu = \operatorname{div}(A(x)\nabla u(x)) = 0. \tag{7}$$

When we say that a solution u satisfies (7) we will always mean, unless otherwise stated, that it does so in the weak, or generalized, sense; i.e. for every $v \in W^{1,2}_0(\Omega)$ we have that

$$\int_\Omega \langle A\nabla u, \nabla v\rangle = 0. \tag{8}$$

A special example is when $A = \text{Id}$ so that

$$Lu = \Delta u = 0. \tag{9}$$

This is Laplace's equation. One can define two important quantities related to this. The first is the Dirichlet energy

$$D(r) := \int_{D_r} |\nabla u(x)|^2 \,\mathrm{d}x. \tag{10}$$

The second is

$$H(r) := \int_{\partial D_r} u^2 \,\mathrm{d}\mathcal{H}^{n-1}, \tag{11}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure on ∂D_r (see Chapter 1, Section 4). Using these two quantities we define the *Almgren frequency function*, for $r \in (0,1)$ and $H(r) \neq 0$, by

$$N(r) := r\frac{D(r)}{H(r)}. \tag{12}$$

In $\mathbb{R}^2$ consider the harmonic functions given in polar co-ordinates by $u_k(r,\theta) = b_k r^k \sin(k\theta)$, where $k \in \mathbb{N}$ and $b_k \in \mathbb{C}$. They are harmonic because they form the imaginary parts of the analytic functions $f_k(z) = b_k z^k$. It is an easy calculation to check that $N(r) = k$ for u_k. That is why it was named the *frequency function*.

Almgren discovered (see [A79]) that $N(r)$ was a non-decreasing function of $r \in (0,1)$. This is one of the theorems this paper will prove. For a general operator of the form (4) however, we want a slight modification of the frequency function. Proving that this modification is also non-decreasing in $r \in (0,1)$ is best done by introducing a Lipschitz Riemannian metric tensor on D_1,

$$\sum_{i,j=1}^{n} g_{ij}(x)\,\mathrm{d}x^i \otimes \mathrm{d}x^j. \tag{13}$$

We will also assume that under a co-ordinate transformation to polar co-ordinates $(r, \theta^1, \ldots, \theta^{n-1})$ this metric tensor takes the form

$$\mathrm{d}r \otimes \mathrm{d}r + r^2 \sum_{i,j=1}^{n-1} b_{ij}(r,\theta)\,\mathrm{d}\theta^i \otimes \mathrm{d}\theta^j. \tag{14}$$

We assume that the b_{ij}'s are real-valued functions on D_1 and that

$$b_{ij}(0,0) = \delta_{ij}, \;\text{ for }\; i,j \in \{1,\ldots,n-1\}, \tag{15}$$

where δ_{ij} is the Kronecker-delta symbol. Additionally we assume that there is a positive constant Λ such that

$$\left|\frac{\partial b_{ij}(r,\theta)}{\partial r}\right| \le \Lambda, \;\text{ for }\; i,j \in \{1,\ldots,n-1\}. \tag{16}$$

On the closed unit ball, B_1, we will assume there is given a real-valued Lipschitz function τ satisfying the following two conditions:

1. There are positive constants A and B such that on B_1 we have:

$$A \le \tau \le B. \tag{17}$$

2. In polar co-ordinates on D_1

$$\tau(0,0) = 1, \;\; \left|\frac{\partial \tau}{\partial r}\right| \le \Lambda \;\text{ a.e.} \tag{18}$$

It is justified to wonder why we are making these assumptions. It turns out (as we will show in Chapter 2, Section 3) that considering weak solutions (in $W^{1,2}(D_1)$) to

$$\mathrm{div}_M(\tau(x)\nabla_M u(x)) = 0, \tag{19}$$

with respect to the metric tensor (13), is equivalent to considering solutions in $W^{1,2}(D_1)$ to

$$Lu = 0. \tag{20}$$

The notation $\text{div}_M X$ denotes the intrinsic divergence of a vector field X on D_1, and $\nabla_M u$ denotes the intrinsic gradient of a function u. Of course, we will have to define a specific form of the metric tensor that will be compatible with (4), (5) and (6), but this will be done in Chapter 2, Section 3. Again, anytime we mention (19) we will always be assuming, unless stated otherwise, that we consider the equation in the weak sense.

Notice how we in (7) look for solutions in $W^{1,2}_{\text{loc}}(\Omega)$ but have in (20) switched to look for solutions in $W^{1,2}(D_1)$. This is because in (3) we assume B_2 is compactly contained in Ω, and so for the sake of simplification we will only deal with solutions in $W^{1,2}(D_1)$.

Under the new assumptions we define the general forms of (10) and (11) via:

Definition 0.0.4. *Let $r \in (0,1)$. Let $\mathrm{d}V_{D_r}$ and $\mathrm{d}V_{\partial D_r}$ be the Riemannian volume elements on D_r and ∂D_r, respectively (as defined in Section 3 of Chapter 1). Define the following:*

$$D(r) := \int_{D_r} \tau(x) |\nabla_M u(x)|^2 \, \mathrm{d}V_{D_r}, \tag{21}$$

$$H(r) := \int_{\partial D_r} \tau(x) u^2(x) \, \mathrm{d}V_{\partial D_r}. \tag{22}$$

Now define the generalized Almgren frequency function *as*

$$N(r) = r \frac{D(r)}{H(r)}, \tag{23}$$

for $H(r) \neq 0$.

We can now state the first theorem that this essay will prove:

Theorem 0.0.5. *If $u \in W^{1,2}(D_1)$ is a non-trivial weak solution of equation (19), then there is a positive constant C depending on n and Λ such that the* modified Almgren frequency function

$$\mathcal{N}(r) := \exp(Cr) N(r) \tag{24}$$

is a non-decreasing function of $r \in (0,1)$.

Finally, we state the last of the main theorems this essay proves:

Theorem 0.0.6. *Let $u \in C^2(\Omega)$ be a non-trivial solution of $\Delta u = 0$ in D_1. Then $N(r)$, as defined in (12), is non-decreasing in $r \in (0,1)$.*

How to read this essay

The level of this essay is intended for a general graduate student in mathematics who has been exposed to the basics of analysis and differential equations theory.

Therefore, we do not assume extensive knowledge of Sobolev spaces, tensor calculus in differential geometry, nor Hausdorff measures. If you have not covered any of these topics before, it is advised that you read the relevant sections of Chapter 1 before proceeding. If you have covered these things however, you can safely skip Chapter 1, but be aware that starting from Chapter 2 we will always be referring to the results of Chapter 1.

Only once you know the material of Chapter 1 will the Introduction and Statement of the Theorems be useful. It provides a concise display of the main theorems this essay will prove, and will be referred to considerably.

Chapter 1 covers, without proofs, the prerequisite knowledge required to understand the subsequent chapters. Section 1 covers L^p and Sobolev spaces, which are used throughout the essay. Section 2 covers some results from elliptic partial differential equations theory from which we can make assumptions that will be crucial for the rest of the essay. Section 3 covers the results from tensor calculus on manifolds required to understand the proofs of Chapter 2. Section 4 covers a very concise account of Hausdorff measures, used mainly to give the reader an intuition of the type of measure we integrate against.

Chapters 2, 3 and 4 provide the proofs of Theorems 0.0.5, 0.0.2 and 0.0.6, respectively. The results of Chapter 2 are needed for Chapter 3, but Chapter 4 can be read in isolation.

Chapter 1

Notation and Preliminaries

1.1 L^p and Sobolev Spaces

This essay will consider the operator L defined in (4) and certain solutions u to $Lu = 0$. Demanding that these solutions be, for example, infinitely differentiable would restrict the type of functions that can solve $Lu = 0$. To remedy this we want to weaken the classical notion of differentiability of functions to allow for more "room" for solutions. A specific space of functions we will consider closely in this essay is the Sobolev space $W^{1,2}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$. In this section we define this space. To this end we must start with the definition of a multi-index.

Everything that follows in this section (unless otherwise stated) can be explored in greater detail in sections 5.1, 5.2 and Appendix A of [E02].

Definition 1.1.1. *Let $u : \Omega \to \mathbb{R}$ be a function, where $\Omega \subseteq \mathbb{R}^n$. Let $x = (x_1, \dots, x_n) \in \Omega$. A* multi-index *$\alpha$ is a vector of the form $\alpha = (\alpha_1, \dots, \alpha_n)$, where each α_i is a non-negative integer. The* order *of the multi-index is $|\alpha| = \sum_{i=1}^{n} \alpha_i$. We define*

$$D^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial {x_1}^{\alpha_1} \dots \partial {x_n}^{\alpha_n}}. \tag{1.1}$$

We define, for k a non-negative integer,

$$D^k u(x) := \{D^\alpha u(x) : |\alpha| = k\}. \tag{1.2}$$

We let $Du(x) := D^1 u(x)$.

Now we define the L^p spaces for $1 \le p \le \infty$.

Definition 1.1.2. *Let $\Omega \subseteq \mathbb{R}^n$, and $1 \le p < \infty$. We define*

$$L^p(\Omega) := \left\{ u : \Omega \to \mathbb{R} \ \ s.t. \ \ \|u\|_{L^p(\Omega)} := \left(\int_\Omega |u|^p \right)^{\frac{1}{p}} < \infty \right\}. \tag{1.3}$$

For $p = \infty$, we define

$$L^\infty(\Omega) := \{u : \Omega \to \mathbb{R} \ \ s.t. \ \ \operatorname*{ess\,sup}_\Omega |u| < \infty\}. \tag{1.4}$$

By essential supremum, $\operatorname*{ess\,sup}_\Omega |u|$, we mean the infimum value of all possible suprema of u taken over $\Omega \backslash Z$ where Z has measure 0. We also define local summability. For $1 \le p \le \infty$ define

$$L^p_{\text{loc}}(\Omega) := \{u : \Omega \to \mathbb{R} \ \ s.t. \ \ u \in L^p(V) \ \ whenever \ V \subset\subset \Omega\}. \tag{1.5}$$

Note that by $V \subset\subset \Omega$ we mean that $V \subset \overline{V} \subset \Omega$ and $\overline{V}$ is compact.

Definition 1.1.3. *Let $1 \le k < \infty$. We define the spaces*

$$C^k(\Omega) = \{u : \Omega \to \mathbb{R} \ \ such\ that \ \ D^\alpha u \ \ exists\ and\ is\ continuous \ \ \forall\, |\alpha| \le k\}, \tag{1.6}$$

$$\begin{aligned} C^k(\overline{\Omega}) = \{u \in C^k(\Omega) : D^\alpha u \ \ &is\ uniformly\ continuous \\ &on\ bounded\ subsets\ of\ \Omega, \ \ \forall\, |\alpha| \le k\}, \end{aligned} \tag{1.7}$$

$$C^\infty(\Omega) = \bigcap_{k=1}^{\infty} C^k(\Omega), \tag{1.8}$$

$$C^\infty(\overline{\Omega}) = \bigcap_{k=1}^{\infty} C^k(\overline{\Omega}), \tag{1.9}$$

$$C^k_{\mathrm{C}}(\Omega) = \{u \in C^k(\Omega) : \ \ u \ vanishes\ outside\ a\ compact\ subset\ of\ \Omega\}. \tag{1.10}$$

A test function *on Ω is a function $v \in C^\infty_{\mathrm{C}}(\Omega)$.*

Now we want to go on to discuss norms on such spaces in order to define the Hölder space $C^{k,\alpha}(\overline{\Omega})$.

Definition 1.1.4. *For bounded and continuous $u : \Omega \to \mathbb{R}$, we define*

$$\|u\|_{C(\overline{\Omega})} := \sup_{\Omega} |u| . \tag{1.11}$$

We define also

$$[u]_{C^{0,\alpha}(\overline{\Omega})} := \sup_{x,y\in\Omega, x\neq y} \left\{ \frac{|u(x) - u(y)|}{|x-y|^{\alpha}} \right\}. \tag{1.12}$$

Now define the α^{th} Hölder norm *as*

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} = \|u\|_{C(\overline{\Omega})} + [u]_{C^{0,\alpha}(\overline{\Omega})}. \tag{1.13}$$

We are in a position to define Hölder spaces now. The Hölder space $C^{k,\alpha}(\overline{\Omega})$ *consists of all functions $u \in C^{k}(\overline{\Omega})$ for which the norm*

$$\|u\|_{C^{k,\alpha}(\overline{\Omega})} := \sum_{|\beta|\leq k} \|D^{\beta}u\|_{C(\overline{\Omega})} + \sum_{|\beta|=k} [D^{\beta}u]_{C^{0,\alpha}(\overline{\Omega})} < \infty. \tag{1.14}$$

Henceforth anytime we mention the space $C^{k,\alpha}(\overline{\Omega})$ we implicitly assume $0 < \alpha < 1$, unless otherwise stated. When we mention $C^{k,\alpha}(\Omega)$ it should be taken to mean approximately the same as $C^{k,\alpha}(\overline{\Omega})$ but where the treatment is local rather than global. We now move on to study weak derivatives.

Definition 1.1.5. *Let u and ϕ be elements of $L^1_{\text{loc}}(\Omega)$ for $\Omega \subseteq \mathbb{R}^n$. Let α be a multi-index of order k. We say that ϕ* is the α^{th} weak partial derivative of u, *and write $D^{\alpha}u = \phi$, if the follwing equation is satisfied for all test functions v:*

$$\int_{\Omega} uD^{\alpha}v = (-1)^k \int_{\Omega} \phi v. \tag{1.15}$$

To really appreciate the subtlety of such a definition, we need to see an example.

Example 1.1.6. Let $\Omega = (-1,1) \subset \mathbb{R}$. Let $\alpha = (1)$. Let u be the function defined on $(-1,1)$ by $u(x) = |x|$. Clearly u fails to be differentiable at $x = 0$. However I claim that the function ϕ, defined by being equal to -1 on $-1 <$

$x < 0$, and 1 on $0 \le x < 1$, is a weak (first) derivative. Indeed, for all test functions v on $(-1, 1)$ we have

$$\begin{aligned}\int_{-1}^{1} u(x)v'(x)\,\mathrm{d}x &= \int_{-1}^{0} -xv'(x)\,\mathrm{d}x + \int_{0}^{1} xv'(x)\,\mathrm{d}x \\ &= \int_{-1}^{0} v(x)\,\mathrm{d}x - \int_{0}^{1} v(x)\,\mathrm{d}x \\ &= -\int_{-1}^{1} \phi(x)v(x)\,\mathrm{d}x,\end{aligned} \tag{1.16}$$

where in the second line we used integration by parts and the fact that $v(-1) = v(1) = 0$.

Example 1.1.6 highlights that we can now have a notion of derivatives for functions which are not differentiable in the classic sense. It also illustrates how Definition 1.1.5 comes about from the integration by parts formula. Indeed, if u is $|\alpha|$ times differentiable in the classic sense, (1.15) is equivalent to the classic integration by parts formula.

Definition 1.1.7. *Let k be a non-negative integer and let $1 \le p \le \infty$. Let $\Omega \subseteq \mathbb{R}^n$. Then the* Sobolev space *$W^{k,p}(\Omega)$ is the space of functions $u \in L^1_{\text{loc}}(\Omega)$ for which for each multi-index α with $|\alpha| \le k$, we have that the α^{th}- weak partial derivative $D^\alpha u$ exists, and belongs to $L^p(\Omega)$.*

We define $W^{k,p}_0(\Omega)$ to be the closure of $C^\infty_C(\Omega)$ in $W^{k,p}(\Omega)$.

1.2 Elliptic Partial Differential Equations Theory

In this section I include some fundamental results from partial differential equations theory that we will utilize in Chapter 2. The first one is Theorem 8.8 in [GT01]. Note that rather than stating the theorems in the most general form as they appear in [GT01], I will state them with our specific operator L in mind.

Theorem 1.2.1 ($W^{2,2}$-elliptic regularity)**.** *Let L be the operator (4) satisfying (5) and (6). Let $\Omega \subset \mathbb{R}^n$ be smooth and connected. For $u \in W^{1,2}(\Omega)$ satisfying (7) we have that $u \in W^{2,2}_{\text{loc}}(\Omega)$.*

The importance of Theorem 1.2.1 is that since we are assuming $B_2 \subset\subset \Omega$, all the solutions we will be dealing with will actually belong to $W^{2,2}(D_1)$. Hence when we take second (weak) derivatives in all the calculations of Chapters 2 and 3, and implicitly assume they belong to $L^2(D_1)$, we are not breaking any rules. The next is Theorem 8.19 from [GT01], useful for us in Chapter 2.

Theorem 1.2.2 (Strong Maximum Principle for Weak Solutions)**.** *Let L be the operator (4) satisfying (5) and (6). Let $\Omega \subset \mathbb{R}^n$ be smooth and connected, and $u \in W^{1,2}(\Omega)$ satisfying (7). If for some ball $b \subset\subset \Omega$ we have that*

$$\sup_b u = \sup_\Omega u \geq 0, \tag{1.17}$$

then u is constant on Ω.

The next theorem ensures that solutions in $W^{1,2}(\Omega)$ to equation (7) are automatically in $C^{0,\alpha}(\bar{b})$ for every $b \subset\subset \Omega$. It is Theorem 8.24 in [GT01].

Theorem 1.2.3 (Interior Hölder Estimate)**.** *Let L be the operator (4) satisfying (5) and (6). Let $\Omega \subset \mathbb{R}^n$ be smooth and connected. Let $u \in W^{1,2}(\Omega)$ satisfying (7). Then for any $b \subset\subset \Omega$, we have that $u \in C^{0,\alpha}(\bar{b})$ for some $\alpha = \alpha(n, \lambda)$.*

The following two theorems, which are Theorem 8.32 and Theorem 8.34 in [GT01], respectively, will ensure, together with Theorem 1.2.3 above, that the solutions we are looking for belong to $C^{1,\alpha}(B_1)$.

Theorem 1.2.4. *Let L be an operator of the type (4) satisfying (5) and (6), with M a bounded domain. Let $u \in C^{1,\alpha}(M)$ satisfy (7). Then for any $b \subset\subset M$ we have that*

$$\|u\|_{C^{1,\alpha}(\bar{b})} \leq C\|u\|_{C(\overline{M})}, \tag{1.18}$$

where C depends on $n, \lambda, \mathcal{K}$ and $\operatorname{dist}(b, \partial M)$*.*

Theorem 1.2.5. *Let S be a $C^{1,\alpha}$ domain and L an operator of the form (4) satisfying (5) and (6). Let $g \in L^\infty(S), f^i \in C^{0,\alpha}(\overline{S})$ and $\Phi \in C^{1,\alpha}(\overline{S})$. Then the generalized Dirichlet problem (in the weak sense)*

$$Lu = g + \sum_{i=1}^{n} \frac{\partial f^i}{\partial x_i} \ \textit{ in } \ S, \tag{1.19}$$

$$u = \Phi \ \textit{ on } \ \partial S, \tag{1.20}$$

is uniquely solvable in $C^{1,\alpha}(\overline{S})$.

Finally, we need a lemma before completing our argument. The proof of it is straightforward by basic calculus.

Lemma 1.2.6. *Let $\Omega \subset \mathbb{R}^n$ be smooth and connected. Let $f \in C^{0,1}(\Omega)$. Then $Df \in L^\infty(\Omega)$.*

Now consider the following argument. Let the S of Theorem 1.2.5 be $D_{\frac{3}{2}}$. For our purposes we do not need to know what a $C^{1,\alpha}$ domain is, but the ball being smooth certainly satisfies this. Let L be the operator in (4) satisfying (5) and (6). Let $g = -\operatorname{div}(A\nabla\chi)u$ and $f^i = (2uA\nabla\chi)_i$, where χ is a smooth cut-off function on $D_{\frac{3}{2}}$ that equals 1 on B_1 and 0 near $\partial D_{\frac{3}{2}}$. Let $\Phi = 0$. Then for solutions $u \in W^{1,2}_{\text{loc}}(\Omega)$ satisfying equation (7) with Ω as in (3), we have by Theorem 1.2.3 that $u \in C^{0,\alpha}(B_{\frac{3}{2}})$, for the α in Theorem 1.2.3.

Using this implies that each f^i is $C^{0,\alpha}(B_{\frac{3}{2}})$, because the a_{ij}'s and $\nabla\chi$ are bounded. Also, note that by Lemma 1.2.6 we have that $g \in L^\infty(D_{\frac{3}{2}})$. Now calculate in the weak sense

$$\begin{aligned}
L(\chi u) &= \operatorname{div}(A\nabla(\chi u)) \\
&= \operatorname{div}(uA\nabla\chi + \chi A\nabla u) \\
&= \operatorname{div}(uA\nabla\chi) + \operatorname{div}(\chi A\nabla u) \\
&= \operatorname{div}(uA\nabla\chi) + \langle\nabla\chi, A\nabla u\rangle + \chi\operatorname{div}(A\nabla u) \\
&= \operatorname{div}(uA\nabla\chi) + \sum_{i=1}^{n}\frac{\partial\chi}{\partial x_i}\left(\sum_{j=1}^{n}a_{ij}\frac{\partial u}{\partial x_j}\right) \\
&= \operatorname{div}(uA\nabla\chi) + \sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\frac{\partial\chi}{\partial x_i}\frac{\partial u}{\partial x_j} \qquad (1.21) \\
&= \operatorname{div}(uA\nabla\chi) + \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n}a_{ij}\frac{\partial\chi}{\partial x_i}u\right) - \sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n}a_{ij}\frac{\partial\chi}{\partial x_i}\right)u \\
&= \operatorname{div}(uA\nabla\chi) + \operatorname{div}(uA\nabla\chi) - \operatorname{div}(A\nabla\chi)u \\
&= 2\operatorname{div}(uA\nabla\chi) - \operatorname{div}(A\nabla\chi)u \\
&= \sum_{i=1}^{n}\frac{\partial f^i}{\partial x_i} + g,
\end{aligned}$$

where we have used that $\operatorname{div}(A\nabla u) = 0$, the fact that A is symmetric, and the product rule. Theorem 1.2.5 now applies and so the generalized problem $L(u\chi) = 0$ is uniquely solvable in $C^{1,\alpha}(B_{\frac{3}{2}})$. Using this in Theorem 1.2.4 implies that $u\chi \in C^{1,\alpha}(B_1)$ and so $u \in C^{1,\alpha}(B_1)$, as χ is just 1 there.

The conclusion of this entire argument is that we can, in the rest of this essay, assume that u is continuously differentiable on B_1 and so whenever we take derivatives of u, or derivatives of some form of its integral on B_1, we know that the procedure is justified without further discussion.

1.3 Tensor Calculus on Manifolds

The aim of this section is to quickly summarize the essentials of tensor calculus on smooth manifolds in order to understand the proofs in this essay. Most of what I will describe in this section is found in [DP10] and in chapters 2 and 3 of [L97], unless otherwise stated. The aim here is to gain an intuition for what tensors are and how we can use them.

In what follows, and for the rest of this essay, we will use Einstein summation convention. For example, a summation on indexed coefficients of the form

$$\sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} A_{\alpha\beta}B_{\beta\gamma}C_{\gamma\delta}, \tag{1.22}$$

will be denoted by

$$A_{\alpha\beta}B_{\beta\gamma}C_{\gamma\delta}. \tag{1.23}$$

In general, we assume that a summation takes place over indices which appear at least twice in the expression, and no summation is assumed over indices that appear once.

Assume for now that we are in $\mathbb{R}^n$ and that we have a (not necessarily orthonormal) system of basis vectors $\{\vec{E}_1, \vec{E}_2, \ldots, \vec{E}_n\}$. Let

$$\vec{E} = \begin{bmatrix} \vec{E}_1 \\ \vec{E}_2 \\ \vdots \\ \vec{E}_n \end{bmatrix} \tag{1.24}$$

Imagine we transform our co-ordinate system linearly:

$$\vec{F} = A\vec{E}, \tag{1.25}$$

where A is some invertible matrix. A straight forward calculation using linear algebra would show that if $\vec{X}$ is a vector in $\mathbb{R}^n$ with components expressed in terms of the first co-ordinate system, and $\vec{V}$ is the corresponding vector in the new co-ordinate system, then

$$\vec{V} = (A^{-1})^T\vec{X}. \tag{1.26}$$

If we had an orthonormal co-ordinate system then $(A^{-1})^T = A$ and so the vectors would vary in “conformity” with the change of co-ordinates. In general this will not be the case, and so normal vectors vary “contrary” to the basis transformation.

However, for certain types of vectors, the transformation is always in “conformity”. Gradient vectors are such an example. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded smooth function. Let the co-ordinate system in $\mathbb{R}^n$ be denoted by $\{x = (x_1, x_2, \ldots, x_n)\}$. With the basis transform described earlier, we have that

$$v_\mu = B_{\mu\gamma} x_\gamma, \tag{1.27}$$

where for ease of notation $B = (A^{-1})^T$ from equation (1.26), and v represents a vector in terms of the new co-ordinates. Let

$$w_\alpha = \frac{\partial f}{\partial x_\alpha}, \tag{1.28}$$

and let

$$z_\alpha = \frac{\partial f}{\partial v_\alpha}. \tag{1.29}$$

Another simple calculation using the chain rule shows that

$$\vec{z} = (B^{-1})^T \vec{w}. \tag{1.30}$$

But $(B^{-1})^T = A$ and therefore we see that gradient vectors vary in "conformity" with the change of co-ordinates. This is why in literature normal vectors are also sometimes called contravariant vectors and gradient vectors are called covariant vectors. When given a co-ordinate chart on a manifold, the components of contravariant vectors are usually upper-indexed: x^γ, whilst those for covariant vectors are lower-indexed: w_α. This makes summation using the Einstein convention very convenient. On a simple vector space however, like $\mathbb{R}^n$, one usually denotes the normal vectors by $X = (X_1, \ldots, X_n)$ and the vectors of the covariant (dual) space by $\omega = (\omega^1, \ldots, \omega^n)$. This also makes Einstein summation more convenient.

It is easily checked that the usual inner product between a contravariant and covariant vector, $\omega^\beta X_\beta$, remains unchanged after a basis transformation. However the same is not true for the inner product between two contravariant vectors, $X_\beta Y_\beta$. But we can modify this to some $g^{\mu\gamma} X_\mu Y_\gamma$ under which taking an inner product becomes invariant under the basis transformations. The object $g^{\mu\gamma}$ is a *covariant 2 -tensor* as it "acts" on two contravariant vectors and yields a real number. We can also have an *l-contravariant tensor* which acts on l covariant vectors and yields a real number.

With this intuition at hand we are now ready to rigorously define tensors on finite dimensional real vector spaces. The notation in the upcoming definitions, including what is upper and lower indexed, is what we will adopt in the remainder of this essay.

Definition 1.3.1. *Let V be a real vector space. Let $V^\star$ denote the dual of V, i.e. the space of covariant vectors. A* k-covariant, l-contravariant tensor*, or a* tensor of type $\binom{k}{l}$*, is a multilinear map*

$$g : V^\star \times \ldots \times V^\star \times V \times \ldots \times V \to \mathbb{R}, \tag{1.31}$$

where the $V^\star$'s appears l times and the V's appear k times.

The space of all tensors of type $\binom{k}{l}$ is denoted by $T^k_l(V)$.

The tensor product *of two tensors $F \in T^k_l(V)$ and $G \in T^r_s(V)$ is $F \otimes G \in T^{k+r}_{l+s}(V)$, defined by:*

$$\begin{aligned} &F \otimes G(\omega^1, \ldots, \omega^{l+s}, X_1, \ldots, X_{k+r}) = \\ &F(\omega^1, \ldots, \omega^l, X_1, \ldots, X_k) G(\omega^{l+1}, \ldots, \omega^{l+s}, X_{k+1}, \ldots, X_{k+r}). \end{aligned} \tag{1.32}$$

If $\{e_1, \ldots, e_n\}$ is a basis set for V, we define the dual basis *for $V^\star$ by $\{\phi^1, \ldots, \phi^n\}$ via $\phi^i(e_j) = \delta_{ij}$.*

The basis elements for the space $T^k_l(V)$ are defined by the tensors of the form

$$e_{j_1} \otimes \ldots \otimes e_{j_l} \otimes \phi^{i_1} \otimes \ldots \otimes \phi^{i_k}, \tag{1.33}$$

where $j_1, \ldots, j_l, i_1, \ldots, i_k \in \{1, \ldots, n\}$. These are defined to act on basis elements of V and $V^\star$ by:

$$e_{j_1} \otimes \ldots \otimes e_{j_l} \otimes \phi^{i_1} \otimes \ldots \otimes \phi^{i_k}(\phi^{s_1}, \ldots, \phi^{s_l}, e_{r_1}, \ldots, e_{r_k}) = \delta_{s_1 j_1} \ldots \delta_{s_l j_l} \delta_{i_1 r_1} \ldots \delta_{i_k r_k}. \tag{1.34}$$

We denote by $\Lambda^k(V)$ the space of k-forms *on V. This is the space of k-covectors that change sign whenever two arguments are interchanged. We define the* wedge product *on 1-forms $\omega^1, \ldots, \omega^k$ by setting*

$$\omega^1 \wedge \ldots \wedge \omega^k(X_1, \ldots, X_k) = \det(\omega^i(X_j)). \tag{1.35}$$

With the above definition at hand any tensor $F \in T^k_l(V)$ can be written in terms of the basis elements by

$$F = F^{j_1 \ldots j_l}_{i_1 \ldots i_k} e_{j_1} \otimes \ldots \otimes e_{j_l} \otimes \phi^{i_1} \otimes \ldots \otimes \phi^{i_k}. \tag{1.36}$$

We want to use our knowledge now to define the Riemannian metric tensor. In general, we would like to have tensors defined on manifolds with some given co-ordinate chart, and not just finite dimensional vector spaces. Knowledge of what exactly a manifold is (and all related information about local co-ordinate charts, partitions of unity, functions on manifolds etc.) will not be fundamental to this essay, and so the precise definitions are omitted. If you would still like to find out, a good source is Chapter 1 of [L03]. For now, think of them as "nice" structures in $\mathbb{R}^n$ with some specified co-ordinate system. Note that because we are limited by our wish not to involve an extensive discussion on manifolds, what follows is just meant for intuition and is not entirely rigorous.

For a manifold M, one can think of the space of tangent vectors at a point p of the manifold, call it T_pM. This is a vector space, and so it would make sense to define a $\binom{k}{l}$-tensor at $p \in M$ to be an element of $T^k_l(T_pM)$.

We will also need to have the intuition of a *bundle of* $\binom{k}{l}$ *tensors on* M, written as

$$T^k_l(M) = \bigsqcup_{p \in M} T^k_l(T_pM). \tag{1.37}$$

It is important to note that the tensor bundle is not just a disjoint union as in (1.37); there is a smooth structure that needs to go along with this, but since this section is meant mainly for intuition, we will omit the details of this. Similarly, we can think of a *bundle of* k*-forms* on M, written as

$$\Lambda^k(M) = \bigsqcup_{p \in M} \Lambda^k(T_pM). \tag{1.38}$$

And finally, we will let TM denote the *tangent bundle* of a smooth manifold M, which can be thought of as a disjoint union (with respect to $p \in M$) of T_pM (where again there is a smooth structure that needs to go with this!)

If $\{x^i\}$ are local co-ordinates on a subset $U \subset M$, and $p \in U$, then the set $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ forms a basis for T_pM, and the basis of the covariant space (the dual space) is $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}$.

A *local frame* $\{e_1, \dots, e_n\}$ for TM are n smooth vector fields defined on some open set $U \subset M$, such that $\{e_1|_p, \dots, e_n|_p\}$ forms a basis for T_pM at each $p \in M$. The *dual coframe*, $\{\phi^1, \dots, \phi^n\}$, are smooth 1-forms satisfying $\phi^i(e_j) = \delta_{ij}$.

A smooth section of some tensor bundle $T^k_l(M)$ is what is known as a *tensor field* on M. The space of $\binom{k}{l}$ tensor fields is denoted by $\mathcal{T}^k_l(M)$. For a co-ordinate frame $\{\frac{\partial}{\partial x^i}\}$ and dual coframe $\{\mathrm{d}x^i\}$, a $\binom{k}{l}$- tensor field F evaluated at a point $p \in M$ has co-ordinate expression

$$F\big|_p = F^{j_1 \dots j_l}_{i_1 \dots i_k}(p) \frac{\partial}{\partial x^{j_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{j_l}} \otimes \mathrm{d}x^{i_1} \otimes \dots \otimes \mathrm{d}x^{i_k}, \tag{1.39}$$

where we use $F\big|_p$ to emphasize that this is where the tensor field is evaluated, in order not to confuse it with the vectors or covectors to which it is applied. We now define the Riemannian metric, adopting definition 9.1.1 of [A]:

Definition 1.3.2. *A* Riemannian metric *on a smooth manifold M is an element of $\mathcal{T}^2(M)$ in the sense that it is a smoothly chosen inner product on each tangent space T_pM, that satisfies the following:*

- *Symmetry: $g(X_p, Y_p) = g(Y_p, X_p)$ for all $X_p, Y_p \in T_pM$, and*
- *Positive semi-definiteness: $g(X_p, X_p) \geq 0$ for all $X_p \in T_pM$, and equality holds if and only if $X_p = 0$.*

By smoothly chosen we mean that for smooth vector fields X and Y, the map $p \to g(X_p, Y_p)$ is smooth.

For this essay we will need the notion of a *Lipschitz metric tensor.* This is simply a Riemannian metric such that when written as

$$g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j, \tag{1.40}$$

for a co-ordinate frame $\{\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\}$ and dual coframe $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}$, then $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ is Lipschitz for all $i, j \in \{1, \dots, n\}$.

I now state five essential derivations that will be very important for the main proofs of this paper. I will give quick remarks on how one can prove some them. All can be found in Chapter 3 of [L97], except Lemma 1.3.8, which I have extracted from page 387 of [H64].

Lemma 1.3.3 (Intrinsic Gradient)**.** *Let M be a smooth manifold on which we have a Riemannian metric g. Let $f : M \to \mathbb{R}$ be a smooth map. Then*

$$\nabla_M f = g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j}, \tag{1.41}$$

where $\nabla_M f$ denotes the intrinsic gradient of f on the manifold, $g^{ij}(x)$ is the inverse of $g_{ij}(x)$, and $\{\frac{\partial}{\partial x^i}\}$ is our co-ordinate frame basis.

Remark 1.3.4. Using the definition of a Riemannian metric notice that we can create covariant vectors from contravariant ones and vice-versa. For instance, we can define $\omega(Y) = g(X,Y)$ for $X,Y \in TM$. Note that ω is covariant as it acts on contravariant vectors. In particular, expanding X in co-ordinates as $X = X^i\frac{\partial}{\partial x^i}$, we get

$$\omega(\cdot) = g(X,\cdot) = g(X^i\frac{\partial}{\partial x^i},\cdot) = X^i g_{ij}\,\mathrm{d}x^j := X_j\,\mathrm{d}x^j. \tag{1.42}$$

This construction of creating a covariant vector is called *lowering an index*. In a similar fashion one can *raise an index* and create contravariant vectors from covariant ones. After having raised an index we usually denote ω by $\omega^{\#}$. The intrinsic gradient of f can then be obtained via $\nabla_M f = \mathrm{d}f^{\#}$. Unpacking the definitions yields Lemma 1.3.3.

Another important point to discuss is the fact that the lemma assumes f is smooth. The types of functions u we will be dealing with however are in $W^{1,2}_{\mathrm{loc}}(\Omega)$ and we would still want to apply the lemma. In order to do so we can use a sequence of mollifiers on u which make it smooth (see [E02], Appendix C, Theorem 6), then insert these smooth functions in (1.41) and pass to a limit. Since Theorem 1.2.4 ensured us that on B_1 the functions u we are considering will be $C^{1,\alpha}$, apply part (iii) of of Theorem 6 in [E02] and we get that Lemma 1.3.3 also applies for $u \in W^{1,2}(D_1)$, which is what we care about.

Now we turn to inner products of tensors, which we will use in Chapter 2.

Lemma 1.3.5 (Tensor Inner Product)**.** *Let $\{x^i\}$ be any local co-ordinate system on a manifold M, and g the Riemannian metric. Let $F,G \in T^k_l(M)$ be expressed as in (1.36). Then the tensor inner product is given by*

$$\langle F,G\rangle = g^{i_1r_1}\dots g^{i_kr_k}g_{j_1s_1}\dots g_{j_ls_l}F^{j_1\dots j_l}_{i_1\dots i_k}G^{s_1\dots s_l}_{r_1\dots r_k}. \tag{1.43}$$

Lemma 1.3.6. *Let M be an oriented n-dimensional manifold, with a Riemannian metric g. Then there exists a unique n - form $\mathrm{d}V$ such that for any oriented orthonormal basis $(e_1,\dots,e_n)$ in T_pM, we have that*

$$\mathrm{d}V(e_1,\dots,e_n) = 1. \tag{1.44}$$

In particular, if $\{e_i\}$ is any oriented local frame with dual coframe $\{\phi^i\}$, then

$$\mathrm{d}V = \sqrt{\det(g_{ij})}\phi^1 \wedge \ldots \wedge \phi^n. \tag{1.45}$$

Remark 1.3.7. The $\mathrm{d}V$ in Lemma 1.3.6 is sometimes called the Riemannian volume element. It is used to define the integral of a function over manifolds. Indeed, if f is a smooth and compactly supported function on an oriented manifold M with Riemannian metric g, then note that $f\,\mathrm{d}V$ is a compactly supported n-form, and so we define the integral of f over M to be

$$\int_M f\,\mathrm{d}V. \tag{1.46}$$

The *volume* of M is defined to be

$$\int_M 1\,\mathrm{d}V. \tag{1.47}$$

Lemma 1.3.8 (Intrinsic Divergence)**.** *Let $X = X^i\frac{\partial}{\partial x^i}$ be a vector field on M. Then the* intrinsic divergence *of X on M is given by the function on M which locally is defined by:*

$$\mathrm{div}_M X = \frac{1}{\sqrt{|\det(g_{ij})|}}\frac{\partial}{\partial x^i}\left(\sqrt{|\det(g_{ij})|}X^i\right). \tag{1.48}$$

Remark 1.3.9. Let M be a smooth oriented manifold with Riemannian metric g and volume element $\mathrm{d}V$ (as described in Lemma 1.3.6). Let X be a vector field on M, and t be a k-form. Consider the $(k-1)$-form $i_X t$ defined by

$$i_X t(V_1, \ldots, V_{k-1}) = t(X, V_1, \ldots, V_{k-1}). \tag{1.49}$$

Then actually the definition of the *intrinsic divergence operator* $\mathrm{div}_M : \mathcal{T}(M) \to C^\infty(M)$ is given by:

$$\mathrm{d}(i_X\,\mathrm{d}V) = (\mathrm{div}_M X)\,\mathrm{d}V, \tag{1.50}$$

where $\mathcal{T}(M)$ denotes the space of smooth sections of TM (the space of smooth vector fields). For details, see Chapter 4 of [C13].

Lemma 1.3.10. *Let $\psi : M \to \mathbb{R}$ be a smooth function, where M is a smooth manifold. Let $X \in \mathcal{T}(M)$. Then*

$$\mathrm{div}_M(\psi X) = \psi\mathrm{div}_M X + \langle\nabla_M\psi, X\rangle. \tag{1.51}$$

Remark 1.3.11. Exactly as in Remark 1.3.4, the lemma can be shown to hold for functions in $W^{1,2}_{\text{loc}}(\Omega)$.

Lemma 1.3.12. *Let M be a bounded and compact oriented manifold. Let $X \in \mathcal{T}(M)$. Let $\hat{n}$ be the outward unit normal to ∂M and let $\mathrm{d}V$ and $\mathrm{d}\hat{V}$ denote the volume elements (as per Lemma 1.3.6) with respect to the Riemannian metric on M and ∂M, respectively. Then*

$$\int_M \mathrm{div}_M X \,\mathrm{d}V = \int_{\partial M} \langle X, \hat{n} \rangle \,\mathrm{d}\hat{V}. \tag{1.52}$$

1.4 Hausdorff measure

We begin with a definition of the Hausdorff measure, taken from [F85].

Definition 1.4.1. *Let β be the Borel σ-algebra on $\mathbb{R}^n$. For any set $\Omega \in \beta$ we define its* diameter *as*

$$\mathrm{diam}(\Omega) = \sup\{|x-y| : x, y \in \Omega\} \in [0,\infty], \;\; \mathrm{diam}(\varnothing) = 0. \tag{1.53}$$

We say that $\{U_i\}_{i\in\mathbb{N}}$ is a δ - cover *of Ω if $\Omega \subseteq \bigcup_{i=1}^{\infty} U_i$ and $0 < \mathrm{diam}(U_i) \leq \delta$ for every $i \in \mathbb{N}$, with $U_i \in \beta$. For $k \in \mathbb{N}$, define*

$$\mathcal{H}^k_\delta(\Omega) = \inf\left\{\sum_{i=1}^{\infty}(\mathrm{diam}(U_i))^k : \{U_i\}_{i\in\mathbb{N}} \;\, \textit{is a}\;\, \delta - \textit{cover for}\;\, \Omega\right\}. \tag{1.54}$$

*We then define the k-*dimensional Hausdorff measure *on Ω as*

$$\mathcal{H}^k(\Omega) = \lim_{\delta\to 0}\mathcal{H}^k_\delta(\Omega). \tag{1.55}$$

Remark 1.4.2.

1. We used Borel σ-algebras on $\mathbb{R}^n$ to define the Hausdorff measure, but of course this could be done for general metric spaces. We will not need this however.

2. For open subsets of $\mathbb{R}^n$, the n-dimensional Hausdorff measure can be thought of as the classic "volume" of the set, and the $(n-1)$-dimensional Hausdorff measure on a surface in $\mathbb{R}^n$ can be thought of as "surface area". When used in integration, $\mathrm{d}\mathcal{H}^n$ will denote the classic "volume element" measure, and $\mathrm{d}\mathcal{H}^{n-1}$ the "area element" measure. That is all the intuition we will need for this essay.

3. Note that we can also write $\mathcal{H}^k(\Omega) = \lim_{\delta \to 0} \mathcal{H}^k_\delta(\Omega) = \sup_{\delta > 0} \mathcal{H}^k_\delta(\Omega)$. This is because as δ becomes smaller the number of δ-covers become more limited and so the infimum, as in Definition 1.4.1, always increases.

Chapter 2

Proving that the modified Almgren frequency function is non-decreasing

In the first section of this chapter, I prove Theorem 0.0.5, using the lemmas in Chapter 1. The proof given in [GL86] is very dense and omits many non-trivial mathematical derivations. Therefore, I will provide the proofs of these and state them as lemmas.

The second section will discuss some important derivations presented in [AKS62], which will be essential in relating Theorem 0.0.5 to our weak solutions $u \in W^{1,2}_{\text{loc}}(\Omega)$ of equation (7). The relationship will be established in the third section.

2.1 Proof of Theorem 0.0.5

Lemma 2.1.1. *Theorem 0.0.5 will be proved if one can show that there is a constant $C = C(n, \Lambda)$ satisfying*

$$C(n, \Lambda) \geq \frac{H'(r)}{H(r)} - \frac{1}{r} - \frac{D'(r)}{D(r)}, \tag{2.1}$$

where D and H are as in (21) and (22), respectively.

Proof. $\mathcal{N}(r)$ is non-decreasing if $\mathcal{N}'(r) \geq 0$. Now, by the quotient rule, for $N(r)$ as in (23),

$$\begin{aligned} N'(r) &= \frac{D(r)H(r) + rD'(r)H(r) - rD(r)H'(r)}{(H(r))^2} \\ &= \frac{D(r)}{H(r)} + \frac{rD'(r)}{H(r)} - \frac{rD(r)H'(r)}{(H(r))^2} \\ &= \frac{N(r)}{r} + \frac{N(r)D'(r)}{D(r)} - \frac{N(r)H'(r)}{H(r)} \\ &= N(r)\left(\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)}\right) \end{aligned} \tag{2.2}$$

and so

$$\begin{aligned} \mathcal{N}'(r) \geq 0 &\iff C\exp(Cr)N(r) + \exp(Cr)N'(r) \geq 0 \\ &\iff C + \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \geq 0, \end{aligned} \tag{2.3}$$

where we obtain the second equivalence of (2.3) by dividing by the positive quantity $\exp(Cr)N(r)$, and using (2.2). The claim follows. □

Lemma 2.1.1 informs us that we need to calculate $H'(r)$ and $D'(r)$. The following lemmas serve this purpose. We start however with some definitions.

Definition 2.1.2. *Let $g^{ij}(x)$ denote the inverse of the matrix $g_{ij}(x)$ presented in equation (13). Let*

$$g(x) := |\det(g_{ij}(x))|. \tag{2.4}$$

Whenever we apply a coordinate transformation into polar co-ordinates, we will abuse notation and write $g(r,\theta)$ and $g_{ij}(r,\theta)$ for the tensors in the new co-ordinates. We will also let

$$b(r,\theta) := |\det(b_{ij}(r,\theta))|, \tag{2.5}$$

where the b_{ij}'s were introduced in (14).

Lemma 2.1.3. *The quantity $H(r)$ as in equation (22) can be written as*

$$H(r) = r^{n-1} \int_{\partial D_1} \tau(r,\theta) u^2(r,\theta) \sqrt{b(r,\theta)}\, \mathrm{d}\theta \tag{2.6}$$

when we transform our co-ordinates into polar co-ordinates (r,θ).

Proof. We have that any point on ∂D_r can be represented in our polar co-ordinates by $(r,\theta) = (r,\theta^1,\ldots,\theta^{n-1})$, where r is fixed as we are on the surface. This is an oriented choice of co-ordinates on ∂D_r, and so by equation (1.45) of Lemma 1.3.6, we have that $\mathrm{d}V_{\partial D_r} = \sqrt{\det(g_{ij}(r,\theta))}\, \mathrm{d}\theta$. Therefore

$$H(r) = \int_{\partial D_r} \tau u^2 \,\mathrm{d}V_{\partial D_r} = \int_{\partial D_1} \tau(r,\theta) u^2(r,\theta) \sqrt{\det(g_{ij}(r,\theta))}\, \mathrm{d}\theta. \tag{2.7}$$

Now $\sqrt{\det(g_{ij}(r,\theta))} = \sqrt{g(r,\theta)}$ by definition. Notice that by equation (14) we obtain:

$$(g_{ij}(r,\theta)) = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & r^2 b_{11}(r,\theta) & \ldots & r^2 b_{1,n-1}(r,\theta) \\ 0 & r^2 b_{21}(r,\theta) & \ldots & r^2 b_{2,n-1}(r,\theta) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & r^2 b_{n-1,1}(r,\theta) & \ldots & r^2 b_{n-1,n-1}(r,\theta) \end{bmatrix}. \tag{2.8}$$

Thus, it is clear that $g(r,\theta) = r^{2(n-1)} |\det((b_{ij}(r,\theta)))| = r^{2(n-1)} b(r,\theta)$, and thus $\sqrt{\det(g_{ij}(r,\theta))} = r^{n-1}\sqrt{b(r,\theta)}$, from which the result follows. □

Now we calculate $H'(r)$. Note that in what follows we might sometimes omit the arguments that the functions take in order to simplify notation.

Lemma 2.1.4. *The derivative of H is given by*

$$H'(r) = \left(\frac{n-1}{r} + O(1) \right) H(r) + 2 \int_{\partial D_r} \tau u u_\rho \,\mathrm{d}V_{\partial D_r}, \tag{2.9}$$

where $O(1)$ is some function of r and θ bounded in absolute value by some constant $C = C(n,\Lambda)$, almost everywhere, and $u_\rho = \langle \nabla_M u, \frac{x}{\rho} \rangle$ is derivative of u in the radial direction (normal to ∂D_r).

Proof. Using Lemma 2.1.3 with the product and chain rule we obtain

$$\begin{aligned} H'(r) &= \frac{\mathrm{d}}{\mathrm{d}r}\left(r^{n-1} \int_{\partial D_1} \tau u^2 \sqrt{b}\, \mathrm{d}\theta \right) \\ &= (n-1) r^{n-2} \int_{\partial D_1} \tau u^2 \sqrt{b}\, \mathrm{d}\theta + r^{n-1} \int_{\partial D_r} \frac{\partial}{\partial \rho}(\tau u^2 \sqrt{b})\, \mathrm{d}\rho\, \mathrm{d}\theta \\ &= \frac{n-1}{r} H(r) + r^{n-1} \int_{\partial D_r} \frac{\partial}{\partial \rho}(\tau u^2 \sqrt{b})\, \mathrm{d}\rho\, \mathrm{d}\theta. \end{aligned} \tag{2.10}$$

Thus

$$\begin{aligned} H'(r) &= \frac{n-1}{r} H(r) + r^{n-1} \int_{\partial D_r} \frac{\partial}{\partial \rho}(\tau \sqrt{b}) u^2\, \mathrm{d}\rho\, \mathrm{d}\theta + r^{n-1} \int_{\partial D_r} \tau \sqrt{b} \frac{\partial}{\partial \rho}(u^2)\, \mathrm{d}\rho\, \mathrm{d}\theta \\ &= \frac{n-1}{r} H(r) + \int_{\partial D_r} \frac{1}{\sqrt{b}} \frac{\partial}{\partial \rho}(\tau \sqrt{b}) u^2\, \mathrm{d}V_{\partial D_r} + 2 \int_{\partial D_r} \tau u u_\rho\, \mathrm{d}V_{\partial D_r}, \end{aligned} \tag{2.11}$$

where we have used that $\mathrm{d}V_{\partial D_r} = r^{n-1}\sqrt{b}\, \mathrm{d}r\, \mathrm{d}\theta$ from the proof of Lemma 2.1.3. Now notice that

$$\frac{1}{\sqrt{b}} \frac{\partial}{\partial \rho}(\tau \sqrt{b}) u^2 = \tau_\rho u^2 + \frac{1}{2b} b_\rho \tau u^2 = \left(\frac{\tau_\rho}{\tau} + \frac{b_\rho}{2b} \right) \tau u^2. \tag{2.12}$$

By (16) and (18), we conclude that τ_ρ and b_ρ are bounded in absolute value by some constant depending on Λ and n, (almost everywhere for τ_ρ), where the n appears because taking a determinant gives us a dimensional dependence. We deduce that b is also automatically bounded (because if on a bounded domain a function has bounded derivative then it must itself be bounded). We also have that τ is bounded from below and above due to (17). Thus the integral of (2.12) on ∂D_r yields $O(1)H(r)$ where $O(1)$ is a function of (r, θ), bounded in absolute value almost everywhere by some constant $C = C(n, \Lambda)$.

Putting all this together we get

$$H'(r) = \left(\frac{n-1}{r} + O(1) \right) H(r) + 2 \int_{\partial D_r} \tau u u_\rho\, \mathrm{d}V_{\partial D_r}, \tag{2.13}$$

as desired.

□

Lemma 2.1.5. *The quantity* $\int_{\partial D_r} \tau u u_\rho \, \mathrm{d}V_{\partial D_r}$ *obtained in (2.13) equals* $D(r)$. *Consequently, we can write*

$$H'(r) = \left(\frac{n-1}{r} + O(1)\right) H(r) + 2D(r). \tag{2.14}$$

Proof. We first use Lemma 1.3.3 to write

$$\nabla_M(u^2) = g^{ij} \frac{\partial(u^2)}{\partial x^i} \frac{\partial}{\partial x^j} = 2u\nabla_M u, \tag{2.15}$$

and then we use Lemma 1.3.10 to write

$$\begin{aligned} \mathrm{div}_M(\tau \nabla_M(u^2)) &= \mathrm{div}_M(2u\tau\nabla_M u) = 2u\mathrm{div}_M(\tau\nabla_M u) + \langle 2\nabla_M u, \tau\nabla_M u\rangle \\ &= 2\tau|\nabla_M u|^2, \end{aligned} \tag{2.16}$$

where we have applied $\mathrm{div}_M(\tau\nabla_M u) = 0$ by equation (19). Thus

$$\int_{D_r} \mathrm{div}_M(\tau\nabla_M(u^2)) \, \mathrm{d}V_{D_r} = 2\int_{D_r} \tau|\nabla_M u|^2 \, \mathrm{d}V_{D_r} = 2D(r). \tag{2.17}$$

On the other hand, Lemma 1.3.12 tells us that

$$\begin{aligned} \int_{D_r} \mathrm{div}_M(\tau\nabla_M(u^2)) \, \mathrm{d}V_{D_r} &= \int_{D_r} \mathrm{div}_M(2u\tau\nabla_M u) \, \mathrm{d}V_{D_r} \\ &= 2\int_{\partial D_r} \langle u\tau\nabla_M u, \frac{x}{\rho}\rangle \, \mathrm{d}V_{\partial D_r} \\ &= 2\int_{\partial D_r} \tau u u_\rho \, \mathrm{d}V_{\partial D_r}. \end{aligned} \tag{2.18}$$

The claim follows. □

Now the aim is to find an expression for $D'(r)$. To this end we start with the following setup:

Fix an r and h between 0 and $\frac{1}{2}$. Define a map $\rho : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ by $\rho(x) = |x|$. Also define, for $t \in \mathbb{R}^+$, a map $w_t : \mathbb{R}^n \to \mathbb{R}^+$ by

$$w_t(x) = \begin{cases} t, & \text{if } \rho(x) \leq r \\ t\frac{r+h-\rho(x)}{h} + \frac{\rho(x)-r}{h}, & \text{if } r \leq \rho(x) \leq r+h \\ 1, & \text{if } \rho(x) \geq r+h \end{cases} \tag{2.19}$$

Now for $0 < t < 1 + \frac{h}{r+h}$, define another map $l_t : \mathbb{R}^n \to \mathbb{R}^n$ by

$$l_t(x) = w_t(x)x. \tag{2.20}$$

For $u \in W^{1,2}(D_1)$ we define $u^t(x) = u(l_t^{-1}(x))$. Also set

$$I[u^t] = \int_{D_1} \tau |\nabla_M(u^t)|^2 \, \mathrm{d}V_{D_1}. \tag{2.21}$$

Lemma 2.1.6. *The map l_t is bi-Lipschitz, and consequently $u^t \in W^{1,2}(D_1)$.*

Proof. We will show that $\exists K \geq 1$ such that

$$\frac{1}{K}|x_1 - x_2| \leq |w_t(x_1)x_1 - w_t(x_2)x_2| \leq K|x_1 - x_2|, \tag{2.22}$$

for all $x_1, x_2 \in \mathbb{R}^n$. Now by (2.19) we see that for all x_1, x_2 with $\rho(x_1), \rho(x_2) \geq r + h$ the claim is clear for all $K \geq 1$. The other cases must be analysed individually. I will do the case $\rho(x_1) \leq r$ and $\rho(x_2) \geq r + h$. The other cases follow a similar logic. We want to show that $\exists K \geq 1$ such that

$$\frac{1}{K}|x_1 - x_2| \leq |tx_1 - x_2| \leq K|x_1 - x_2|, \tag{2.23}$$

for all $x_1, x_2 \in \mathbb{R}^n$, with $\rho(x_1) \leq r$ and $\rho(x_2) \geq r + h$. First note that

$$\begin{aligned} |x_1 - x_2| &= |x_1 - tx_1 + tx_1 - x_2| \\ &\leq |tx_1 - x_2| + |1 - t||x_1| \\ &\leq |tx_1 - x_2| + |1 - t|r \\ &\leq \left(1 + \frac{|1-t|r}{r + h - rt}\right)|tx_1 - x_2|, \end{aligned} \tag{2.24}$$

where in order to get the last inequality we first note that since $x_1 \in B_r$ and $x_2 \in B_{r+h}^C$ we can deduce that $|tx_1 - x_2| \geq r + h - rt$. Then note that $r + h - rt > r + h - r(1 + \frac{h}{r+h}) = \frac{h^2}{r+h} > 0$, so that $\frac{|tx_1 - x_2|}{r+h-rt} \geq 1$. This gives us a first bound. For the second we get that

$$\begin{aligned} |tx_1 - x_2| &= |tx_1 + tx_2 - tx_2 - x_2| \\ &\leq t|x_1 - x_2| + |t - 1||x_2| \\ &= t|x_1 - x_2| + |t - 1||x_2 - x_1 + x_1| \\ &\leq (t + |t - 1|)|x_1 - x_2| + r|t - 1| \\ &\leq \left(t + |t - 1| + \frac{r|t-1|}{h}\right)|x_1 - x_2|, \end{aligned} \tag{2.25}$$

where in the last inequality we used the fact that $|x_1 - x_2| \geq h$. Putting all this together and choosing a maximal constant proves this special case. The general case is similar, and it follows easily that $u^t \in W^{1,2}(D_1)$.

□

Lemma 2.1.7. *For* $u \in W^{1,2}(D_1)$ *solving* $\mathrm{div}_M(\tau \nabla_M u) = 0$ *in* D_1*, it follows that*

$$\frac{\mathrm{d}}{\mathrm{d}t} I[u^t]\bigg|_{t=1} = 0. \tag{2.26}$$

Proof. This is an important concept, and so I want to discuss the origins of this result and how it can be generalized. Assume for now that we are looking for a solution $u \in C^2(\overline{\Omega})$ to the problem $\Delta u = 0$ (we are in the usual Cartesian co-ordinates). Assume that Ω is as in (3). Define the functional

$$E(u) = \int_{\Omega} |\nabla u|^2 . \tag{2.27}$$

It is simply checked that if $\Delta u = 0$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u + t\psi)\bigg|_{t=0} = 0 \tag{2.28}$$

for all $\psi \in C_{\mathrm{C}}^{\infty}(\Omega)$ (i.e. u is a critical point of E), and for all $w \in C^2(\overline{\Omega})$ with $w|_{\partial\Omega} = u|_{\partial\Omega}$, we have that

$$E(u) \leq E(w). \tag{2.29}$$

This is known as *Dirichlet's Principle* (see [H97], Theorem 1); harmonic functions minimize the Dirichlet energy. However, for our purposes, we want a similar statement for solutions in $W^{1,2}_{\mathrm{loc}}(\Omega)$. Theorem 6 in [H97] tells us that for $w \in W^{1,2}(\Omega)$, there exists a unique $u \in W^{1,2}(\Omega)$ with the same trace as w on the boundary such that this u minimizes (2.27) amongst all functions that have the same trace as w on $\partial\Omega$.

And indeed, we may define a functional (now with respect to the Riemannian metric (13)) via

$$I(u) = \int_{D_1} \tau \, |\nabla_M u|^2 \, \mathrm{d}V_{D_1}. \tag{2.30}$$

From the same ideas as above, we can show that I has a minimizer amongst functions in $W^{1,2}(\Omega)$ with fixed boundary data, and by Theorem 1.2.2, it can be shown that if u satisfies equation (19), then u is the unique minimizer amongst all functions with the same boundary data.

The radial deformation introduced in (2.19) serves the same purpose. Notice that for $t = 1$, $w_1(x) = 1$ and so $l_1(x) = x$ and so $u^1(x) = u(x)$. Thus amongst all perturbations near $t = 1$ the minimizer of I is at $t = 1$, and that is why equation (2.26) follows. Notice that the radial deformation introduced fixes the boundary data, because when $x \in \partial D_1$, $\rho(x) \geq r + h$ and so $w_t(x) = 1$ and $u^t(x) = u(x)$. □

Now

$$\begin{aligned} I[u^t] &= \int_{D_1} \tau |\nabla_M(u^t)|^2 \, \mathrm{d}V_{D_1} \\ &= \int_{D_{rt}} \tau |\nabla_M(u^t)|^2 \, \mathrm{d}V_{D_{rt}} + \int_{D_{r+h}\setminus D_{rt}} \tau |\nabla_M(u^t)|^2 \, \mathrm{d}V_{D_{r+h}\setminus D_{rt}} + \int_{D_1\setminus D_{r+h}} \tau |\nabla_M(u^t)|^2 \, \mathrm{d}V_{D_1\setminus D_{r+h}} \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{2.31}$$

We now compute $\frac{\mathrm{d}I_1}{\mathrm{d}t}\Big|_{t=1}, \frac{\mathrm{d}I_2}{\mathrm{d}t}\Big|_{t=1}, \frac{\mathrm{d}I_3}{\mathrm{d}t}\Big|_{t=1}$.

Lemma 2.1.8. *We have that*

$$\frac{\mathrm{d}I_1}{\mathrm{d}t}\Big|_{t=1} = (n - 2 + O(r))D(r), \tag{2.32}$$

where $O(r)$ denotes a function of the polar co-ordinates r and θ which is bounded in absolute value almost everywhere by a constant Cr, where C is some constant depending on Λ and n.

Proof. First we want to calculate $|\nabla_M(u^t)|^2$, when $x \in D_{rt}$. Now $u^t(x) = u(l_t^{-1}(x))$ for $x \in D_{rt}$. So we are looking for y such that $l_t(y) = w_t(y)y = x \in D_{rt}$. Note if $\rho(y) \leq r$ then $w_t(y) = t$ and so $ty = x \in D_{rt}$ is satisfied. Therefore $y = \frac{x}{t}$. Thus

$$u^t(x) = u\left(\frac{x}{t}\right). \tag{2.33}$$

Now by Lemma 1.3.3 we have that

$$\nabla_M(u^t(x)) = g^{ij} \frac{\partial u^t}{\partial x^i} \frac{\partial}{\partial x^j} := F^j \frac{\partial}{\partial x^j}. \tag{2.34}$$

Now use Lemma 1.3.5 for this tensor of type $\binom{0}{1}$, to get

$$\begin{aligned}|\nabla_M(u^t)|^2 &= \langle \nabla_M(u^t), \nabla_M(u^t)\rangle \\ &= g_{mn}F^m F^n \\ &= g_{mn}g^{im}\frac{\partial u^t}{\partial x^i}g^{kn}\frac{\partial u^t}{\partial x^k} \\ &= \delta_{in}\frac{\partial u^t}{\partial x^i}g^{kn}\frac{\partial u^t}{\partial x^k} \\ &= g^{kn}\frac{\partial u^t}{\partial x^k}\frac{\partial u^t}{\partial x^n}.\end{aligned} \tag{2.35}$$

Now of course since our metric tensor is given in polar co-ordinates (ρ, θ), satisfying the form (13), we have that $g^{11}(\rho,\theta) = 1$ and $g^{ij}(\rho,\theta) = \rho^{-2}b^{i-1,j-1}(\rho,\theta)$ for $i, j \in \{2, \ldots, n\}$, and where $(b_{ij})^{-1} = (b^{ij})$. To show the latter statement invert the matrix in (2.8). Thus

$$\int\limits_{D_{rt}} \tau|\nabla_M(u^t)|^2 \,\mathrm{d}V_{D_{rt}} = \int\limits_0^{rt}\int\limits_{S^{n-1}} \tau(\rho,\theta)g^{kn}(\rho,\theta)\frac{\partial}{\partial x^k}u\left(\frac{\rho}{t},\theta\right)\frac{\partial}{\partial x^n}u\left(\frac{\rho}{t},\theta\right)\mathrm{d}V_{D_{rt}} \tag{2.36}$$

where $x^1 = \rho$ and $x^i = \theta^{i-1}$ for $i = 2, 3, \ldots, n$. Now by Lemma 1.3.6, we have that $\mathrm{d}V_{D_{rt}} = \sqrt{g(\rho,\theta)}\,\mathrm{d}\rho\,\mathrm{d}\theta$. Write $s = \frac{\rho}{t}$. Putting this together and using the chain rule on the partial derivatives in (2.36), we obtain the expression

$$\begin{aligned}\int\limits_{D_{rt}} \tau|\nabla_M(u^t)|^2 \,\mathrm{d}V_{D_{rt}} = &\int\limits_0^{r}\int\limits_{S^{n-1}} \tau(st,\theta)u_s^2(s,\theta)\frac{\sqrt{g(st,\theta)}}{t}\,\mathrm{d}s\,\mathrm{d}\theta + \\ &\int\limits_0^{r}\int\limits_{S^{n-1}} t\tau(st,\theta)u_{\theta_i}(s,\theta)u_{\theta_j}(s,\theta)b^{ij}(st,\theta)\sqrt{g(st,\theta)}\,\mathrm{d}s\,\mathrm{d}\theta.\end{aligned} \tag{2.37}$$

The next step is to estimate $\frac{\sqrt{g(st,\theta)}}{t}$ and $tb^{ij}(st,\theta)\sqrt{g(st,\theta)}$. For this purpose let

$$\sqrt{b(s,\theta)} = 1 + \epsilon(s,\theta), \quad (1+\epsilon(s,\theta))b_{ij}(s,\theta) = \delta_{ij} + \overline{\epsilon_{ij}}(s,\theta), \tag{2.38}$$

where $\epsilon(0,0) = 0$ and $\overline{\epsilon_{ij}}(0,0) = 0$. Seeing that $b(0,0) = 1$, we have that ϵ represents an error function of how much the square root of the determinant of (b_{ij}) differs from 1. Likewise, note that $(1+\epsilon)b_{ij} = (1+\epsilon)(\delta_{ij}+\gamma_{ij}) = \delta_{ij} + \epsilon\delta_{ij} + \gamma_{ij} + \epsilon\gamma_{ij}$, and so we have simply chosen to call $\overline{\epsilon_{ij}} = \epsilon\delta_{ij} + \gamma_{ij} + \epsilon\gamma_{ij}$, where (γ_{ij}) measures the error of (b_{ij}) being the identity matrix.

Now use (2.38) together with the fact that $\sqrt{g(st,\theta)} = (st)^{n-1}\sqrt{b(st,\theta)}$ from the proof of Lemma 2.1.3 to conclude that

$$\sqrt{g(st,\theta)} = (st)^{n-1}(1+\epsilon(st,\theta)), \tag{2.39}$$

and the fact that going from $b^{ij}(st,\theta)$ to $b_{ij}(st,\theta)$ picks up the factor $(st)^{-2}$ to conclude that

$$b^{ij}(st,\theta)\sqrt{g(st,\theta)} = (st)^{n-3}(\delta_{ij} + \overline{\epsilon_{ij}}(st,\theta)). \tag{2.40}$$

We now want to plug (2.39) and (2.40) into (2.37), differentiate with respect to t and evaluate the expression at $t=1$. When we differentiate on ϵ and $\overline{\epsilon_{ij}}$ we will use the chain rule to get factors of s out of the expressions and then differentiate with respect to the radial variable ρ. It follows from (2.38) that $\epsilon_\rho = (\sqrt{b})_\rho$. Using (16) we get that $|\epsilon_\rho|$ is bounded by a constant depending on Λ (coming from the b_{ij}'s) and n (coming from the fact that taking a determinant will give us a dimensional dependence). We clearly see that the same thing applies to $\overline{\epsilon_{ij}}$. Using this fact in (2.37) we get equation (2.32), as desired.

□

Lemma 2.1.9. *We have that*

$$\left.\frac{\mathrm{d}I_2}{\mathrm{d}t}\right|_{t=1} = 2r\int_{\partial D_r} \tau u_\rho^2 \,\mathrm{d}V_{\partial D_r} - r\int_{\partial D_r} \tau|\nabla_M u|^2 \,\mathrm{d}V_{\partial D_r} \tag{2.41}$$

Proof. The proof is very similar to the case above, but now $x \in D_{r+h}\backslash D_{rt}$ and at the end of the calculation you take the limit as $h \to 0^+$, to get the boundary integral.

□

Lemma 2.1.10. *We have that* $\left.\frac{\mathrm{d}I_3}{\mathrm{d}t}\right|_{t=1} = 0$.

Proof. For $x \in D_1\backslash D_{r+h}$ we have that $\rho(x) \geq r+h$ and so $l_t^{-1}(x) = x$, and so $u^t(x) = u(x)$. Hence I_3 is independent of t and the result follows. □

Putting Lemmas 2.1.7, 2.1.8, 2.1.9, and 2.1.10 together we get that

$$(n-2+O(r))D(r)+2r\int\limits_{\partial D_r}\tau u_\rho^2\,\mathrm{d}V_{\partial D_r}-r\int\limits_{\partial D_r}\tau|\nabla_M u|^2\,\mathrm{d}V_{\partial D_r}=0, \tag{2.42}$$

where we recognize:

$$\int\limits_{\partial D_r}\tau|\nabla_M u|^2\,\mathrm{d}V_{\partial D_r}=D'(r). \tag{2.43}$$

Thus we have

$$rD'(r)-(n-2+O(r))D(r)=2r\int\limits_{\partial D_r}\tau u_\rho^2\,\mathrm{d}V_{\partial D_r}. \tag{2.44}$$

From this we prove Theorem 0.0.5:

Proof of Theorem 0.0.5. We have that

$$\begin{aligned}\frac{1}{r}+\frac{D'(r)}{D(r)}-\frac{H'(r)}{H(r)}&=\frac{1}{r}+\frac{n-2+O(r)}{r}+\frac{2}{D(r)}\int\limits_{\partial D_r}\tau u_\rho^2\,\mathrm{d}V_{\partial D_r}-\frac{H'(r)}{H(r)}\\&=\frac{n-1}{r}+\frac{O(r)}{r}+\frac{2}{D(r)}\int\limits_{\partial D_r}\tau u_\rho^2\,\mathrm{d}V_{\partial D_r}\\&\quad-\frac{n-1}{r}-O(1)-2\frac{\int_{\partial D_r}\tau u u_\rho\,\mathrm{d}V_{\partial D_r}}{H(r)}\\&=O(1)+2\frac{\int_{\partial D_r}\tau u_\rho^2\,\mathrm{d}V_{\partial D_r}}{\int_{\partial D_r}\tau u u_\rho\,\mathrm{d}V_{\partial D_r}}-2\frac{\int_{\partial D_r}\tau u u_\rho\,\mathrm{d}V_{\partial D_r}}{\int_{\partial D_r}\tau u^2\,\mathrm{d}V_{\partial D_r}},\end{aligned} \tag{2.45}$$

where in the first equality we used (2.44), in the second (2.13), and in the last (22) and Lemma 2.1.5. Now since u and its (weak) derivative are in $L^2(D_1)$ and τ is bounded, we can use the Cauchy-Schwarz inequality to conclude that

$$\langle\sqrt{\tau}u,\sqrt{\tau}u_\rho\rangle^2\le\|\sqrt{\tau}u\|_2^2\|\sqrt{\tau}u_\rho\|_2^2. \tag{2.46}$$

This implies that

$$2\frac{\int\limits_{\partial D_r} \tau u_\rho^2 \, \mathrm{d}V_{\partial D_r}}{\int\limits_{\partial D_r} \tau u u_\rho \, \mathrm{d}V_{\partial D_r}} - 2\frac{\int\limits_{\partial D_r} \tau u u_\rho \, \mathrm{d}V_{\partial D_r}}{\int\limits_{\partial D_r} \tau u^2 \, \mathrm{d}V_{\partial D_r}} \geq 0, \tag{2.47}$$

so that

$$\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \geq O(1) \geq -C(n, \Lambda). \tag{2.48}$$

By Lemma 2.1.1, Theorem 0.0.5 is now proved. □

2.2 Some results from [AKS62]

Now I explore some of the results presented in sections 2 and 3 of [AKS62]. This section is independent of the work we have done thus far, but everything will be tied up neatly in section 3 of this chapter. I commence with the assumptions made in order for Theorem 2 in [AKS62] to hold.

- For some $r_0 > 0$, the manifold D_{2r_0} is endowed with a Lipschitz metric tensor $G_{ij}(x)$.
- There are positive constants k_1 and k_2 such that for all $\zeta \in \mathbb{R}^n$, and all $x \in D_{2r_0}$, we have
$$k_1|\zeta|^2 \leq G_{ij}(x)\zeta^i\zeta^j \leq k_2|\zeta|^2. \tag{2.49}$$
- There is a positive constant k such that for all $\zeta \in \mathbb{R}^n$ and $x \in D_{2r_0}$ we have that
$$\left|\frac{\partial G_{ij}}{\partial e}\zeta^i\zeta^j\right| \leq k|\zeta|^2 \text{ a.e.} \tag{2.50}$$
Here $\frac{\partial}{\partial e}$ is the radial derivative in D_{2r_0}.

We then define a function $r : D_{2r_0} \to \mathbb{R}$ via

$$r(x) := \sqrt{G_{ij}(0)x^i x^j}, \tag{2.51}$$

and a new metric tensor on D_{2r_0}

$$\overline{G_{ij}}(x) := G_{ij}(x)\left(G^{kl}(x)\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^l}\right), \tag{2.52}$$

where of course $G^{ij}(x)$ denotes the inverse of $G_{ij}(x)$. In the proof of Theorem 2 in [AKS62] the authors show that for

$$\overline{k_1} := \frac{k_1^2}{k_2}, \quad \overline{k_2} := \frac{k_2^2}{k_1}, \tag{2.53}$$

we have that for all $\zeta \in \mathbb{R}^n$ and $x \in D_{2r_0}$

$$\overline{k_1}|\zeta|^2 \leq \overline{G_{ij}}(x)\zeta^i\zeta^j \leq \overline{k_2}|\zeta|^2. \tag{2.54}$$

Also, almost everywhere on D_{2r_0} and for all $\zeta \in \mathbb{R}^n$, we have that

$$\left|\frac{\partial \overline{G_{ij}}}{\partial e}\zeta^i\zeta^j\right| \leq \overline{k}|\zeta|^2, \tag{2.55}$$

where

$$\overline{k} = \frac{6kk_2^2}{k_1^2}\sqrt{\frac{k_2}{k_1}}. \tag{2.56}$$

With this at hand, section 3 of [AKS62] shows that in the metric $\overline{G_{ij}}$ the function r is actually the geodesic distance from 0 to x. To show this consider the following system of ODE's:

$$\frac{\mathrm{d}x^i}{\mathrm{d}\sigma} = \overline{G^{ij}}\frac{\partial r}{\partial x^j}, \quad i = 1, 2, \dots n. \tag{2.57}$$

Here $\overline{G^{ij}}(x)$ denotes the inverse of $\overline{G_{ij}}(x)$. Consider the ellipsoid

$$S := \{r(x) < q := r_0\sqrt{k_1}\}. \tag{2.58}$$

It can be checked that S is contained in D_{r_0}. Let Σ denote the boundary of S. Let t be a point on Σ and consider (2.57) together with an initial condition $x(q) = t$. The ODE can now be solved for a unique solution $x(\sigma)$. Now notice that

$$\begin{aligned}\frac{\mathrm{d}r(x(\sigma))}{\mathrm{d}\sigma} &= \frac{\partial r}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}\sigma} \\ &= \overline{G^{ij}}\frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j} \\ &= G^{ij}(x)\left(G^{kl}(x)\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^l}\right)^{-1}\frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j} \\ &= 1,\end{aligned} \tag{2.59}$$

where in the second line we used equation (2.57) and in the third line equation (2.52). We conclude that $r(x(\sigma)) = \sigma + c$, but recall that $q = r(t) = r(x(q))$ so c must be 0.

Notice that at the origin, the right hand side of (2.57) fails to be continuous as $\frac{\partial r}{\partial x^j}$ fails to be continuous there due to (2.51). Thus we only consider our solutions $x(\sigma) \in (0, q]$.

Now our solution $r(x(\sigma)) = \sigma$ means that for points t on Σ we get a system of simple arcs joining 0 to t, mutually disjoint and filling out S.

The next step is to introduce polar co-ordinates $(t^1, \dots, t^{n-1}, r)$ on Σ, where $r = r(x)$. Note that by the chain rule

$$\overline{G_{ij}}\, \mathrm{d}x^i\, \mathrm{d}x^j = \overline{G_{ij}} \left(\frac{\partial x^i}{\partial t^\alpha} \mathrm{d}t^\alpha + \frac{\partial x^i}{\partial r} \mathrm{d}r \right) \left(\frac{\partial x^j}{\partial t^\beta} \mathrm{d}t^\beta + \frac{\partial x^j}{\partial r} \mathrm{d}r \right), \tag{2.60}$$

where α and β run from 1 to $n-1$. Expanding (2.60), and using (2.57) and (2.59) we get that the terms involving the partial derivative with respect to r satisfy

$$\overline{G_{ij}} \frac{\partial x^i}{\partial r} \frac{\partial x^j}{\partial r} = \overline{G_{ij}}\, \overline{G^{ik}} \frac{\partial r}{\partial x^k} \overline{G^{jl}} \frac{\partial r}{\partial x^l} = \overline{G^{kl}} \frac{\partial r}{\partial x^k} \frac{\partial r}{\partial x^l} = 1, \tag{2.61}$$

whilst those that have one partial derivative with respect to r and another with respect to t^α satisfy

$$\overline{G_{ij}} \frac{\partial x^i}{\partial t^\alpha} \frac{\partial x^j}{\partial r} = \overline{G_{ij}} \frac{\partial x^i}{\partial t^\alpha} \overline{G^{jl}} \frac{\partial r}{\partial x^l} = \frac{\partial x^l}{\partial t^\alpha} \frac{\partial r}{\partial x^l} = \frac{\partial r}{\partial t^\alpha} = 0. \tag{2.62}$$

Putting all this together we can conclude that our metric tensor takes the form

$$\overline{G_{ij}}\, \mathrm{d}x^i\, \mathrm{d}x^j = (\mathrm{d}r)^2 + r^2 B_{\alpha\beta}\, \mathrm{d}t^\alpha\, \mathrm{d}t^\beta, \tag{2.63}$$

where

$$B_{\alpha\beta} = \frac{1}{r^2} \overline{G_{ij}} \frac{\partial x^i}{\partial t^\alpha} \frac{\partial x^j}{\partial t^\beta}. \tag{2.64}$$

From these derivations it can thereafter be shown that the polar co-ordinates we have chosen are geodesic relative to the metric $\overline{G_{ij}}$, and hence r is the geodesic distance from 0 to x. Furthermore, it can be deduced by considering a point $t \in \Sigma$ and a contravariant vector T tangential to Σ, that as a function of r, $B_{\alpha\beta}$ satisfies

$$\left| \frac{\partial B_{\alpha\beta}}{\partial r} \right| \le \omega, \tag{2.65}$$

with

$$\omega = \frac{30 k k_2^5 \sqrt{k_2}}{k_1^7}. \tag{2.66}$$

2.3 Tying it all up

Now we want to use the discussion of section 2.2 in order to provide a theorem like 0.0.5, but for weak solutions $u \in W^{1,2}_{\text{loc}}(\Omega)$ of equation (7); which was our original equation of interest. We substitute in the following:

Let $r_0 = \frac{1}{2}$, where r_0 is as defined in section 2.2 above, so that on $D_1 \subset \mathbb{R}^n$ we define a Lipschitz Riemannian metric by

$$G_{ij}(x)\,\mathrm{d}x^i \otimes \mathrm{d}x^j = a^{ij}(x)(\det A)^{\frac{1}{n-2}}\,\mathrm{d}x^i \otimes \mathrm{d}x^j, \tag{2.67}$$

where $a^{ij}(x)$ denote entries of $A(x)^{-1}$, the inverse of the matrix defined for (4). Now recall that (6) told us that A had lowest eigenvalue λ and highest $\frac{1}{\lambda}$ on Ω. By writing $A = P^{-1}DP$ where D is the matrix of eigenvalues of A it is easy to check that $(\det A)^{\frac{1}{n-2}} A^{-1}$ has lowest eigenvalue a constant depending on λ and n (the n coming from taking a determinant) and a highest eigenvalue a bigger constant also depending on λ and n. Translating this into equation (2.49), we have that the k_1 and k_2 appearing depend on λ and n.

Also, note that the Lipschitz condition in (5) implies that partial derivatives of a_{ij} are bounded in absolute value by $\mathcal{K}$, almost everywhere. Translating this into the partial derivative defined in (2.50), we notice that we need to take partial derivatives of $a_{ij}^{-1}(\det A)^{\frac{1}{n-2}}$. To do this we will use the chain rule and product rule, and exactly because the a_{ij}'s are bounded in absolute value from below (due to (6)), the final outcome will be bounded in absolute value almost everywhere by a constant that depends on n (coming from the determinant), λ (coming from the a_{ij} bounds), and $\mathcal{K}$ (coming from the partial derivative that will act on a_{ij}). Hence it follows that the k in (2.50) depends on n, λ and $\mathcal{K}$.

We set $(G^{ij}(x)) = (G_{ij}(x))^{-1}$, and we define the function $r : D_1 \to \mathbb{R}$ via

$$r(x) = \sqrt{G_{ij}(0)x^i x^j}. \tag{2.68}$$

With this at hand we can introduce a new Riemannian metric

$$\overline{G_{ij}}(x)\,\mathrm{d}x^i \otimes \mathrm{d}x^j, \tag{2.69}$$

where $\overline{G_{ij}}(x) \coloneqq G_{ij}(x)\left(G^{kl}(x)\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^l}\right)$, just as in (2.52).

Lemma 2.3.1. *$Lu = 0$ in D_1 if and only if $\operatorname{div}_M(\nabla_M u) = 0$ in the Riemannian metric defined by equation (2.67).*

Proof. The following equivalences should be considered in the weak sense, where $G = |\det((G_{ij}))|$:

$$\begin{aligned}
\operatorname{div}_M(\nabla_M u) = 0 &\iff \frac{1}{\sqrt{G}}\frac{\partial}{\partial x^k}\left(\sqrt{G}G^{ik}\frac{\partial u}{\partial x^i}\right) = 0 \\
&\iff \frac{\partial}{\partial x^k}\left(\sqrt{G}G^{ik}\frac{\partial u}{\partial x^i}\right) = 0 \\
&\iff \frac{\partial}{\partial x^k}\left(a_{ik}\frac{\partial u}{\partial x^i}\right) = 0 \\
&\iff \operatorname{div}(A\nabla u) = 0 \\
&\iff Lu = 0,
\end{aligned} \tag{2.70}$$

where in the first equivalence we used Lemmas 1.3.3 and 1.3.8, in the fourth equivalence we used that A is symmetric, and in the third equivalence we computed:

$$\begin{aligned}
\sqrt{G}G^{ik} &= \sqrt{\det(G_{ij})}G^{ik} \\
&= \sqrt{\det\left((a^{ij})(\det A)^{\frac{1}{n-2}}\right)}\left((a^{ik})(\det A)^{\frac{1}{n-2}}\right)^{-1} \\
&= \sqrt{\det(A^{-1})(\det A)^{\frac{n}{n-2}}}\,a_{ik}(\det A)^{\frac{-1}{n-2}} \\
&= (\det A)^{\frac{1}{n-2}}a_{ik}(\det A)^{\frac{-1}{n-2}} \\
&= a_{ik}.
\end{aligned} \tag{2.71}$$

□

Now I proved in the section above that in intrinsic geodesic polar co-ordinates with pole at (0,0), we can write the metric (2.69) as

$$\mathrm{d}r \otimes \mathrm{d}r + r^2 b_{ij}(r,\theta)\,\mathrm{d}\theta^i \otimes \mathrm{d}\theta^j, \tag{2.72}$$

where according to (2.65) and (2.66) the b_{ij}'s satisfy (16), where the constant Λ in (16) depends on $\omega = \omega(k, k_1, k_2)$ but we have already established that k, k_1 and k_2 depend on $n, \mathcal{K}$ and λ. Finally, by (2.72) we can write

$$\operatorname{div}_M(\nabla_M u) = 0 \iff \operatorname{div}_{\tilde{M}}(\tau\nabla_{\tilde{M}} u) = 0, \tag{2.73}$$

where we use M to denote the manifold endowed with the Riemannian metric defined in (2.67), and $\tilde{M}$ to denote the one endowed with the metric defined by (2.72), and where τ is Lipschitz on D_1 satisfying conditions (17) and (18) where the constants in these conditions all depend on Λ but which in turn depends on $n, \mathcal{K}$ and λ. Putting Theorem 0.0.5 together with Lemma 2.3.1 and condition (2.73), we obtain the following theorem:

Theorem 2.3.2. *Let Ω be a smooth, bounded and connected subset of $\mathbb{R}^n$ compactly containing B_2, where $n \in \mathbb{N}, n \geq 3$. Consider an elliptic operator acting on u via $Lu = \operatorname{div}(A\nabla u)$, where A is an $n \times n$ matrix satisfying conditions (5) and (6). If $Lu = 0$ weakly, then there is a constant $C = C(n, \lambda, \mathcal{K})$ such that $\mathcal{N}(r)$, as defined in (24), but this time with respect to the Riemannian metric (2.69), is a non-decreasing function of $r \in (0,1)$.*

Chapter 3

Proving strong unique continuation

The aim of this section is to provide the proof for Theorem 0.0.2. To do so we must first prove Theorem 0.0.3, as we will need the doubling condition in our proof. The proofs are based on those in [GL86], but here I prove the steps that the paper leaves out. We commence with an important remark.

Remark 3.0.1. If we go back to the proof of Theorem 0.0.5, we notice that we could have shown that the modified Almgren frequency function was non-decreasing in $r \in (0,1)$ by using balls centred at $D_r(x_0)$ for any $x_0 \in \Omega$ as long as $B_{2r}(x_0) \subset\subset \Omega$, so that we do not hit the boundary. The choice $x_0 = 0$ was just for convenience. Hence it follows that the constant $C = C(n, \lambda, \mathcal{K})$ in Theorem 2.3.2 works relative to any such ball $D_r(x_0)$.

The proof below for the doubling condition will therefore without loss of generality be given for $D_R(0)$ for $0 < R < \frac{1}{2}$ (the case $R = 0$ is trivially true), but note that the same conclusion will hold for any ball $D_R(x_0)$, $0 < R < \frac{1}{2}$ as long as $B_{2R}(x_0) \subset\subset \Omega$.

Proof of Theorem 0.0.3. Assume the conditions stated in Theorem 0.0.3. Since $\operatorname{div}(A\nabla u) = 0$ applies in Ω, we have that Theorem 2.3.2 applies. I.e. the function $\mathcal{N}(r)$ given is non-decreasing in $r \in (0,1)$.

Let H be defined as in (22), where τ is as in (2.73). Notice that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left(\log\frac{H(r)}{r^{n-1}}\right) &= \left(\frac{H'(r)r^{n-1}-(n-1)H(r)r^{n-2}}{r^{2(n-1)}}\right)\frac{r^{n-1}}{H(r)}\\ &= \frac{H'(r)}{H(r)}-\frac{n-1}{r}.\end{aligned} \tag{3.1}$$

Now by (2.14) we can write

$$\frac{H'(r)}{H(r)}-\frac{n-1}{r} = O(1)+2\frac{D(r)}{H(r)} = O(1)+\frac{2\mathcal{N}(r)\exp(-Cr)}{r}, \tag{3.2}$$

where in the last equality we used the definition of $\mathcal{N}(r)$. Note that the C that appears depends on $n,\mathcal{K}$ and λ according to Theorem 2.3.2. Now for $0<R<\frac{1}{2}$, integrating (3.1) from R to $2R$ yields

$$\begin{aligned}\log\frac{H(2R)}{(2R)^{n-1}}-\log\frac{H(R)}{R^{n-1}} = \log\left(\frac{H(2R)}{H(R)2^{n-1}}\right) &= \int_R^{2R}\left(O(1)+\frac{2\mathcal{N}(r)\exp(-Cr)}{r}\right)\mathrm{d}r\\ &\le C'R+2\mathcal{N}(1)\int_R^{2R}\frac{1}{r}\,\mathrm{d}r\\ &= C'R+2\mathcal{N}(1)\log(2),\end{aligned} \tag{3.3}$$

where C' is a constant that depends on n and Λ, but as discussed in Chapter 2, Λ depends on n,λ and $\mathcal{K}$. In the second line we used the fact that $\mathcal{N}(r)$ is non-decreasing in $r\in(0,1)$ to conclude that $\mathcal{N}(r)\le\mathcal{N}(1)$, and that $\exp(-Cr)$ is decreasing on $(0,1)$ to conclude $\exp(-Cr)\le 1$. Exponentiating now yields

$$\begin{aligned}\frac{H(2R)}{H(R)2^{n-1}} &\le \exp(C'R)\exp(2\mathcal{N}(1)\log(2))\\ &= \exp(C'R)(4^{\mathcal{N}(1)})\\ &= D',\end{aligned} \tag{3.4}$$

where D' is a constant depending on $n,\mathcal{K},\lambda$ and u. The u dependence comes from the fact that $\mathcal{N}(1)$ depends on u. Thus

$$\begin{aligned}&H(2R)\le 2^{n-1}D'H(R) \implies\\ &\int_{\partial D_{2R}}\tau u^2\,\mathrm{d}V_{\partial D_{2R}} \le 2^{n-1}D'\int_{\partial D_R}\tau u^2\,\mathrm{d}V_{\partial D_R}.\end{aligned} \tag{3.5}$$

Integrating in R now gives

$$\int_{D_{2R}} \tau u^2 \, \mathrm{d}V_{D_{2R}} \leq 2^{n-1} D' \int_{D_R} \tau u^2 \, \mathrm{d}V_{D_R} = C(n, \lambda, \mathcal{K}, u) \int_{D_R} \tau u^2 \, \mathrm{d}V_{D_R}. \tag{3.6}$$

Divide by the positive and bounded τ to complete the proof. □

Now we prove strong unique continuation, but before that we make another important remark:

Remark 3.0.2.

1. Assume u vanishes to infinite order at $x_0 \in \Omega$. Definition 0.0.1 then applies and so for a sufficiently small ball $D_\delta(x_0)$ and for all natural numbers j we have that $\int_{D_\delta(x_0)} u^2 = \mathcal{O}(\delta^j)$.
2. Assume that we show that $u \equiv 0$ on $D_\delta(x_0)$.
3. Pick a point $y \in D_\delta(x_0)$ near the boundary of the ball. Enclose it with the ball $D_\gamma(y)$ such that $D_\gamma(y) \subset D_\delta(x_0)$. Then $u \equiv 0$ on $D_\gamma(y)$ also.
4. Recall from Remark 3.0.1 that the conclusion of the doubling condition holds with respect to any ball as long as the radius is less that $\frac{1}{2}$ and that twice the ball is compactly contained in Ω. I.e. the radius is independent of the solution function u. In particular it holds for $D_\gamma(y)$ as long as $\gamma < \frac{1}{2}$ and $D_{2\gamma}(y) \subset\subset \Omega$.
5. By the doubling condition, we have that $\int_{D_{2\gamma}(y)} u^2 \leq C(n, \lambda, \mathcal{K}, u) \int_{D_\gamma(y)} u^2 = C \times 0 = 0$. Hence even though the constant C of the doubling condition depends on u, it is multiplied by 0 and so the u dependence is wiped out. And hence u is identically 0 on $D_{2\gamma}(y)$.
6. Now take a union $D_\delta(x_0) \cup D_{2\gamma}(y)$, on which u is identically 0. We pick a point $y_1 \in D_\delta(x_0) \cup D_{2\gamma}(y)$ that is near the boundary and repeat the same thing, until we have "patched up" Ω.

Proof of Theorem 0.0.2. Assume without loss of generality that u, which solves $\operatorname{div}(A\nabla u) = 0$ weakly, vanishes to infinite order at $0 \in \Omega$. I.e. for $\delta > 0$ small enough, we have that $\int_{D_\delta} u^2 = O(\delta^N)$ for every $N \in \mathbb{N}$. Notice that this will also hold for all $\delta_0 \leq \delta$. We will show that $u \equiv 0$ on D_δ.

Let $\beta \in \mathbb{R}$, to be chosen. Using the doubling condition we get that

$$\begin{aligned}\int_{D_\delta} u^2 \, \mathrm{d}V_{D_\delta} &\le C^k \int_{D_{\delta 2^{-k}}} u^2 \, \mathrm{d}V_{D_{\delta 2^{-k}}} \\ &= C^k |D_{\delta 2^{-k}}|^\beta \frac{1}{|D_{\delta 2^{-k}}|^\beta} \int_{D_{\delta 2^{-k}}} u^2 \, \mathrm{d}V_{D_{\delta 2^{-k}}} \\ &= C^k \omega_n^\beta (\delta^n 2^{-kn})^\beta \frac{1}{|D_{\delta 2^{-k}}|^\beta} \int_{D_{\delta 2^{-k}}} u^2 \, \mathrm{d}V_{D_{\delta 2^{-k}}},\end{aligned} \tag{3.7}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. Now choose β such that $C 2^{-n\beta} = 1$, and note that we obtain

$$\int_{D_\delta} u^2 \, \mathrm{d}V_{D_\delta} \le (\omega_n \delta^n)^\beta \frac{1}{|D_{\delta 2^{-k}}|^\beta} \int_{D_{\delta 2^{-k}}} u^2 \, \mathrm{d}V_{D_{\delta 2^{-k}}}. \tag{3.8}$$

Since u vanishes of infinite order at $x = 0$, taking a limit as $k \to \infty$ yields

$$\int_{D_\delta} u^2 \, \mathrm{d}V_{D_\delta} \le \lim_{k \to \infty} (\omega_n \delta^n)^\beta \frac{1}{|D_{\delta 2^{-k}}|^\beta} \int_{D_{\delta 2^{-k}}} u^2 \, \mathrm{d}V_{D_{\delta 2^{-k}}} = 0, \tag{3.9}$$

so that $u \equiv 0$ on D_δ. This completes the proof. □

Chapter 4

A non-geometric approach to the Almgren frequency function and some corollaries

In the rest of this chapter we assume that our operator $L = \Delta$ and that $Lu = 0$ (no longer in the weak sense) for solutions u belonging to $C^2(\Omega)$, where Ω satisfies condition (3). For non-trivial solutions u we set the Almgren frequency function to be

$$N(r) := r\frac{D(r)}{H(r)}, \tag{4.1}$$

where

$$D(r) := \int_{D_r} |\nabla u|^2 \tag{4.2}$$

and

$$H(r) := \int_{\partial D_r} u^2, \tag{4.3}$$

where $H(r) \neq 0$. We prove something very similar to Theorem 0.0.5, using the ideas from [HL], section 2.2. Before that however, we need a couple of lemmas.

Lemma 4.0.1. *Let $u_\rho = \langle \nabla u, \frac{x}{\rho}\rangle$ denote the normal derivative of u on ∂D_r. Then*

$$H'(r) = \frac{n-1}{r}H(r) + 2\int_{\partial D_r} uu_\rho. \tag{4.4}$$

Proof. We have

$$\begin{aligned} H(r) &= \int_{\partial D_r} u^2(x)\,\mathrm{d}\mathcal{H}^{n-1}(x) \\ &= r^{n-1}\int_{|y|=1} u^2(ry)\,\mathrm{d}\mathcal{H}^{n-1}(y), \end{aligned} \tag{4.5}$$

and therefore, by the product rule,

$$\begin{aligned} H'(r) &= (n-1)r^{n-2}\int_{|y|=1} u^2(ry)\,\mathrm{d}\mathcal{H}^{n-1}(y) + 2\int_{\partial D_r} uu_\rho \\ &= \frac{n-1}{r}H(r) + 2\int_{\partial D_r} uu_\rho, \end{aligned} \tag{4.6}$$

as desired.

□

Lemma 4.0.2. *Let $D(r)$ be as in (4.2). Then*

$$D'(r) = \frac{n-2}{r}D(r) + 2\int_{\partial D_r} u_\rho^2. \tag{4.7}$$

Proof. First recall that

$$D'(r) = \int_{\partial D_r} |\nabla u|^2\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.8}$$

Now if we denote (in $\mathbb{R}^n$) $x = (x_1, \dots, x_n)$, note that for $x \in \partial D_r$, we have

$$\frac{1}{r}\sum_{i=1}^n \frac{x_i^2}{r}|\nabla u|^2 = |\nabla u|^2. \tag{4.9}$$

So (4.8) can be written as

$$D'(r) = \frac{1}{r}\int_{\partial D_r} \sum_{i=1}^n \frac{x_i^2}{r}|\nabla u|^2\,\mathrm{d}\mathcal{H}^{n-1} = \frac{1}{r}\sum_{i=1}^n \int_{\partial D_r} x_i|\nabla u|^2\frac{x_i}{r}\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.10}$$

We have written it in such a form in order to apply integration by parts:

$$\int_{D_r} \frac{\partial}{\partial x_i}(x_i|\nabla u|^2)\,\mathrm{d}\mathcal{H}^n = \int_{\partial D_r} x_i|\nabla u|^2 \frac{x_i}{r}\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.11}$$

Using the product rule we compute:

$$\begin{aligned}\int_{D_r} \frac{\partial}{\partial x_i}(x_i|\nabla u|^2)\,\mathrm{d}\mathcal{H}^n &= \int_{D_r} |\nabla u|^2\,\mathrm{d}\mathcal{H}^n + \int_{D_r} x_i \frac{\partial}{\partial x_i}\left(\sum_{k=1}^n \left(\frac{\partial u}{\partial x_k}\right)^2\right)\mathrm{d}\mathcal{H}^n \\ &= \int_{D_r} |\nabla u|^2\,\mathrm{d}\mathcal{H}^n + \int_{D_r} 2x_i \left(\sum_{k=1}^n \frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_i \partial x_k}\right)\mathrm{d}\mathcal{H}^n \qquad (4.12) \\ &= \int_{D_r} |\nabla u|^2\,\mathrm{d}\mathcal{H}^n + 2\sum_{k=1}^n \int_{D_r} x_i \frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_k \partial x_i}\,\mathrm{d}\mathcal{H}^n.\end{aligned}$$

Note that in the very last term of (4.12) we used Clairaut's theorem (symmetry of second derivatives theorem) to switch the order of differentiation, valid because $u \in C^2(\Omega)$.

Now denote $n = \frac{x}{\rho}$ on ∂D_r, so that $n_i = \frac{x_i}{\rho}$. Using integration by parts on the second term of (4.12) we obtain:

$$\begin{aligned}\int_{D_r} x_i \frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_k \partial x_i}\,\mathrm{d}\mathcal{H}^n &= \int_{\partial D_r} x_i \frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i} n_k\,\mathrm{d}\mathcal{H}^{n-1} - \int_{D_r}\left(\delta_{ik}\frac{\partial u}{\partial x_k} + x_i \frac{\partial^2 u}{\partial x_k^2}\right)\frac{\partial u}{\partial x_i}\,\mathrm{d}\mathcal{H}^n \\ &= r\int_{\partial D_r} n_i \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k} n_k\,\mathrm{d}\mathcal{H}^{n-1} - \int_{D_r}\left(\delta_{ik}\frac{\partial u}{\partial x_k} + x_i \frac{\partial^2 u}{\partial x_k^2}\right)\frac{\partial u}{\partial x_i}\,\mathrm{d}\mathcal{H}^n.\end{aligned} \tag{4.13}$$

Now we would like to sum (4.13) over k. Note that $\sum_{k=1}^n \frac{\partial u}{\partial x_k} n_k = u_\rho$, $\sum_{k=1}^n \delta_{ik}\frac{\partial u}{\partial x_k}\frac{\partial u}{\partial x_i} = (\frac{\partial u}{\partial x_i})^2$, and $\sum_{k=1}^n x_i \frac{\partial^2 u}{\partial x_k^2}\frac{\partial u}{\partial x_i} = x_i(\Delta u)\frac{\partial u}{\partial x_i} = 0$. Putting this together, summing (4.13) gives

$$\sum_{k=1}^n \int_{D_r} x_i \frac{\partial u}{\partial x_k}\frac{\partial^2 u}{\partial x_k \partial x_i}\,\mathrm{d}\mathcal{H}^n = r\int_{\partial D_r} n_i \frac{\partial u}{\partial x_i} u_\rho\,\mathrm{d}\mathcal{H}^{n-1} - \int_{D_r}\left(\frac{\partial u}{\partial x_i}\right)^2\mathrm{d}\mathcal{H}^n. \tag{4.14}$$

Hence (4.12) becomes

$$\int_{D_r} \frac{\partial}{\partial x_i}(x_i|\nabla u|^2)\,\mathrm{d}\mathcal{H}^n = \int_{D_r} |\nabla u|^2\,\mathrm{d}\mathcal{H}^n + 2r\int_{\partial D_r} n_i \frac{\partial u}{\partial x_i} u_\rho\,\mathrm{d}\mathcal{H}^{n-1} - 2\int_{D_r}\left(\frac{\partial u}{\partial x_i}\right)^2 \mathrm{d}\mathcal{H}^n. \tag{4.15}$$

Because of (4.10) we now need to also sum (4.15) over i, yielding

$$\begin{aligned}\sum_{i=1}^n \int_{D_r} \frac{\partial}{\partial x_i}\left(x_i|\nabla u|^2\right)\mathrm{d}\mathcal{H}^n &= n\int_{D_r} |\nabla u|^2\,\mathrm{d}\mathcal{H}^n + 2r\int_{\partial D_r} u_\rho^2\,\mathrm{d}\mathcal{H}^{n-1} - 2\int_{D_r} |\nabla u|^2\,\mathrm{d}\mathcal{H}^n \\ &= (n-2)\int_{D_r} |\nabla u|^2\,\mathrm{d}\mathcal{H}^n + 2r\int_{\partial D_r} u_\rho^2\,\mathrm{d}\mathcal{H}^{n-1} \\ &= (n-2)D(r) + 2r\int_{\partial D_r} u_\rho^2\,\mathrm{d}\mathcal{H}^{n-1}.\end{aligned} \tag{4.16}$$

And now finally using (4.16) in (4.10) we obtain

$$D'(r) = \frac{n-2}{r}D(r) + 2\int_{\partial D_r} u_\rho^2\,\mathrm{d}\mathcal{H}^{n-1}. \tag{4.17}$$

□

Lemma 4.0.3. *The quantity $D(r)$ can also be written as*

$$D(r) = \int_{\partial D_r} u u_\rho. \tag{4.18}$$

Proof. Note that by the product rule we have that

$$\Delta u^2 = \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(2u\frac{\partial u}{\partial x_i}\right) = 2|\nabla u|^2 + 2u\Delta u = 2|\nabla u|^2. \tag{4.19}$$

And hence

$$D(r) = \int_{D_r} |\nabla u|^2 = \frac{1}{2}\int_{D_r} \Delta u^2 = \frac{1}{2}\int_{D_r} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial(u^2)}{\partial x_i}\right) = \int_{\partial D_r} u u_\rho. \tag{4.20}$$

□

Proof of Theorem 0.0.6. We want to show $N'(r) \geq 0$. Using (4.1) and the quotient rule we get

$$\begin{aligned} N'(r) &= \frac{(D(r) + rD'(r))H(r) - rD(r)H'(r)}{H(r)^2} \\ &= \frac{D(r)}{H(r)} + \frac{rD'(r)}{H(r)} - \frac{rD(r)}{H(r)}\frac{H'(r)}{H(r)} \\ &= \frac{N(r)}{r} + N(r)\frac{D'(r)}{D(r)} - N(r)\frac{H'(r)}{H(r)} \\ &= N(r)\left(\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)}\right). \end{aligned} \tag{4.21}$$

As $N(r)$ is non-negative, all we need to show is that $\frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} \geq 0$. By Lemmas 4.0.1, 4.0.2 and 4.0.3, we have

$$\begin{aligned} \frac{1}{r} + \frac{D'(r)}{D(r)} - \frac{H'(r)}{H(r)} &= \frac{1}{r} + \frac{n-2}{r} + 2\frac{\int\limits_{\partial D_r} u_\rho^2}{\int\limits_{\partial D_r} uu_\rho} - \frac{n-1}{r} - 2\frac{\int\limits_{\partial D_r} uu_\rho}{\int\limits_{\partial D_r} u^2} \\ &= 2\left(\frac{\int\limits_{\partial D_r} u_\rho^2}{\int\limits_{\partial D_r} uu_\rho} - \frac{\int\limits_{\partial D_r} uu_\rho}{\int\limits_{\partial D_r} u^2}\right). \end{aligned} \tag{4.22}$$

If we show that $\left(\int\limits_{\partial D_r} uu_\rho\right)^2 \leq \left(\int\limits_{\partial D_r} u^2\right)\left(\int\limits_{\partial D_r} u_\rho^2\right)$ then we're done. And indeed, the Cauchy-Schwarz inequality tells us that $|\langle u, u_\rho\rangle|^2 \leq \|u\|_2^2 \|u_\rho\|_2^2$ on ∂D_r, which is the inequality desired. □

We now discuss some corollaries of Theorem 0.0.6. We begin by a corollary that states that the surface integral of u^2 on a large sphere is controlled by the surface integral on a smaller sphere.

Corollary 4.0.4. *Let $u \in C^2(\Omega)$ be a non-trivial solution to $\Delta u = 0$. Then for any $0 < R_1 \leq R_2 \leq 1$ we have that*

$$\frac{H(R_2)}{H(R_1)} \leq \left(\frac{R_2}{R_1}\right)^{n-1+2N(R_2)}. \tag{4.23}$$

Proof. The claim is obvious if $0 < R_1 = R_2$. So assume $R_1 < R_2$. Lemmas 4.0.1 and 4.0.3 imply that

$$H'(r) = \frac{n-1}{r} H(r) + 2D(r), \tag{4.24}$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}r} \log(H(r)) = \frac{H'(r)}{H(r)} = \frac{n-1}{r} + \frac{2}{r} N(r). \tag{4.25}$$

Now integrate (4.25) with respect to r between R_1 and R_2, to obtain

$$\begin{aligned} \log H(R_2) - \log H(R_1) = \log\left(\frac{H(R_2)}{H(R_1)}\right) &= (n-1)\log\left(\frac{R_2}{R_1}\right) + 2\int_{R_1}^{R_2} \frac{N(r)}{r}\,\mathrm{d}r \\ &= \log\left(\left(\frac{R_2}{R_1}\right)^{n-1}\right) + 2\int_{R_1}^{R_2} \frac{N(r)}{r}\,\mathrm{d}r \end{aligned} \tag{4.26}$$

Now exponentiate in (4.26), and use Theorem 0.0.6 to get

$$\begin{aligned} \frac{H(R_2)}{H(R_1)} &= \left(\frac{R_2}{R_1}\right)^{n-1} \exp\left(2\int_{R_1}^{R_2} \frac{N(r)}{r}\,\mathrm{d}r\right) \\ &\leq \left(\frac{R_2}{R_1}\right)^{n-1} \exp\left(2N(R_2)\log\left(\frac{R_2}{R_1}\right)\right) \\ &= \left(\frac{R_2}{R_1}\right)^{n-1} \exp\left(\log\left(\left(\frac{R_2}{R_1}\right)^{2N(R_2)}\right)\right) \\ &= \left(\frac{R_2}{R_1}\right)^{n-1} \left(\frac{R_2}{R_1}\right)^{2N(R_2)} \\ &= \left(\frac{R_2}{R_1}\right)^{n-1+2N(R_2)}, \end{aligned} \tag{4.27}$$

which is the result we desire. □

The importance of this corollary is also that it allows us to easily prove a very general form of the doubling condition (cf. Theorem 0.0.3) for harmonic functions on D_1. We state this result as the next corollary.

Corollary 4.0.5. *Let $u \in C^2(\Omega)$ be a non-trivial solution to $\Delta u = 0$. Then for any $R \in (0, \frac{1}{2})$ and any $S \in [1, 2]$ we have that*

$$\int_{D_{SR}} u^2 \leq S^{2N(1)+n-1} \int_{D_R} u^2. \tag{4.28}$$

In particular, the case $S = 2$ gives the doubling condition (cf. Theorem 0.0.3) for harmonic functions.

Proof. Substitute $R_1 = R$ and $R_2 = SR$ in Corollary 4.0.4. We obtain

$$\frac{H(SR)}{H(R)} \leq S^{n-1+2N(SR)} \leq S^{n-1+2N(1)}, \tag{4.29}$$

because the frequency function is non-decreasing. Using the definition of H and re-arranging, this implies

$$\int_{\partial D_{SR}} u^2 \leq S^{n-1+2N(1)} \int_{\partial D_R} u^2. \tag{4.30}$$

Now integrate (4.30) from 0 to R to get the volume integral (4.28) as required. □

We now prove that the volume integral of u^2 on D_1 is can be bounded from below and above by terms that relate to its surface integral on ∂D_1.

Corollary 4.0.6. *Let $u \in C^2(\Omega)$ be a non-trivial solution to $\Delta u = 0$. Then*

$$\frac{1}{2N(1)+n} \int_{\partial D_1} u^2 \leq \int_{D_1} u^2 \leq \frac{1}{n} \int_{\partial D_1} u^2. \tag{4.31}$$

Proof. Let's start with the upper bound. Let $R \in (0, 1)$. We will substitute $R_2 = 1$ and $R_1 = R$ in the first line of equation (4.27) to write

$$\frac{H(1)}{H(R)} = \frac{1}{R^{n-1}} \exp\left(2 \int_R^1 \frac{N(r)}{r} \, \mathrm{d}r\right) \geq \frac{1}{R^{n-1}}, \tag{4.32}$$

as $2\int_R^1 \frac{N(r)}{r}\,\mathrm{d}r \geq 0$. Now notice that

$$\int_{D_1} u^2 = \int_0^1 \int_{\partial D_R} u^2 = \int_0^1 H(R)\,\mathrm{d}R \leq H(1)\int_0^1 R^{n-1}\,\mathrm{d}R = \frac{H(1)}{n}, \tag{4.33}$$

which is the upper bound required, by the definition of H.

As for the lower bound we have by equation (4.32) and Theorem 0.0.6 that

$$\frac{H(1)}{H(R)} = \frac{1}{R^{n-1}}\exp\left(2\int_R^1 \frac{N(r)}{r}\,\mathrm{d}r\right) \leq \frac{1}{R^{n-1}}\exp\left(2N(1)\log\left(\frac{1}{R}\right)\right) = \frac{1}{R^{2N(1)+n-1}}. \tag{4.34}$$

Thus

$$\int_0^1 H(R)\,\mathrm{d}R \geq H(1)\int_0^1 R^{2N(1)+n-1}\,\mathrm{d}R = \frac{H(1)}{2N(1)+n}, \tag{4.35}$$

which is the lower bound desired, by the definition of H. □

Notice that equality can indeed happen in (4.31), just take $u = 1$. Hence this is in fact an optimal inequality.

Our final corollary will tell us that the limit of the frequency function as r approaches 0 from above is actually the order of vanishing of u at 0. First we need to define however the order of vanishing of a function. We adopt the definition found in [Z14].

Definition 4.0.7. *Let $u : \Omega \to \mathbb{R}$ be a function, where $\Omega \subseteq \mathbb{R}^n$. It's* order of vanishing *at $x_0 \in \Omega$ is k if*

$$D^\alpha u(x_0) = 0, \tag{4.36}$$

for all multi-indices α with $|\alpha| < k$.

Now that we have the definition, we prove the following:

Corollary 4.0.8. *Let $u \in C^2(\Omega)$ be a non-trivial solution to $\Delta u = 0$. Let k be the order of vanishing of u at 0. Then*

$$\lim_{r\to 0^+} N(r) = k. \tag{4.37}$$

Proof. First, by Theorem 0.0.2 we know that u does not vanish to infinite order at 0 (as otherwise u would be identically 0). Also, $\lim_{r\to 0^+} N(r)$ exists as $N(r)$ is non-increasing as $r \to 0^+$, and bounded below by 0.

Now if u is harmonic then u is also real analytic (for the proof, see Theorem 1.28 in [ABR01]). Hence, we can express u by a Taylor series. Now since $D^\alpha u(0) = 0$ for all $|\alpha| < k$, we have that the term of smallest degree in the Taylor expansion of u has degree k. So we can write

$$u(x) = P(x) + R(x), \tag{4.38}$$

where P is a homogenous polynomial of degree k, and R is the remainder polynomial where the term of smallest degree has degree at least $k+1$. Now $\Delta P = 0$, because if not, then there must be some terms in ΔR which cancel away the the non-zero terms of ΔP (as $\Delta u = 0$), which means there is at least one term in R of degree k, which is a contradiction to what we have assumed. Hence both ΔP and ΔR equal 0. Now

$$N(r) = \frac{r \int_{D_r} |\nabla(P+R)|^2}{\int_{\partial D_r} (P+R)^2} = \frac{r\left(\int_{D_r} |\nabla P|^2 + 2\int_{D_r} |\nabla P||\nabla R| + \int_{D_r} |\nabla R|^2\right)}{\int_{\partial D_r} P^2 + 2\int_{\partial D_r} PR + \int_{\partial D_r} R^2}. \tag{4.39}$$

Notice that if we factor out the term involving the ∇P from the numerator of (4.39), and the term involving the P in the denominator, and then take a limit as $r \to 0^+$, all the terms remaining that involve an R or ∇R will vanish, because the term of smallest degree in R is of higher degree than the degree of P. Hence we obtain

$$\lim_{r\to 0^+} N(r) = \lim_{r\to 0^+} \frac{r \int_{D_r} |\nabla P|^2}{\int_{\partial D_r} P^2}. \tag{4.40}$$

Now since P is a homogenous harmonic polynomial of degree k, it forms the real or imaginary part of an analytic function of the form that allows us to write in polar co-ordinates $P(x) = r^k \chi(\theta)$, where χ is the restriction of P to the surface of the $(n-1)$-dimensional hypersphere. Recall that

$D(r) = \int\limits_{D_r} |\nabla P|^2 = \int\limits_{\partial D_r} P P_\rho$, by Lemma 4.0.3. Clearly $P_\rho = k r^{k-1} \chi(\theta)$, so that

$$\frac{r \int\limits_{D_r} |\nabla P|^2}{\int\limits_{\partial D_r} P^2} = \frac{r \int\limits_{\partial D_r} k r^{2k-1} \chi^2(\theta)}{\int\limits_{\partial D_r} r^{2k} \chi^2(\theta)} = k, \tag{4.41}$$

and the result follows. □

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