Extra Questions - MTH4104 Tutorial 1 - Complex Numbers and Polynomials

Ali Raad
e-mail: a.i.raad@qmul.ac.uk

Queen Mary University of London

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The questions marked with a (*) are a little more tricky, but also more fun.

1. (a) Let $f(z) = z^{2018} - 1$. By recalling that $1 = e^{2\pi ki}$ for all $k \in \mathbb{Z}$, find the roots of f.

   (b) (*) Describe, in terms of the roots of f, the roots of the polynomial $g(z) = z^{2017} + z^{2016} + \ldots + z + 1$.

2. Compute $\sqrt{\frac{1}{2} + i\frac{\sqrt{3}}{2}}$, expressing your answer as $a + ib$.

3. Prove that a polynomial of degree 3 with real coefficients (i.e. one which looks like $f(z) = az^3 + bz^2 + cz + d$, where $a, b, c, d \in \mathbb{R}$ $a \neq 0$), always has a real root by:

   (a) Considering the behaviour of f for very large positive and very large negative real values of z, and appealing to the Intermediate Value Theorem (look it up on Wikipedia if you do not know it).

   (b) Assuming it has one complex, non-real root, and thinking about what the other roots must look like.

4. Recall that a linear subspace S of a complex vector space V is a subset $S \subset V$ where $a + b \in S$ whenever $a, b \in S$ and $ks \in S$ whenever $k \in \mathbb{C}$ and $s \in S$. Note that $C(\mathbb{C}) := \{f : \mathbb{C} \to \mathbb{C} : f \text{ is continous}\}$ is a complex vector space with pointwise addition and scalar multiplication. Show that the space S of all polynomials with complex coefficients is a linear subspace of $C(\mathbb{C})$.

5. (*) Let f be a polynomial with complex coefficients, of degree 2016. Assume $f(z) = \frac{1}{z}$ for all $z \in \{1, 2, \ldots 2017\}$. Find $f(2018)$.

6. Is there a polynomial f with complex coefficients such that $f(n) = (-1)^n$ for all $n \in \mathbb{N}$. If yes, find it. If not, prove that there isn't.

7. Let $f(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$ be a polynomial with complex coefficients, of even degree n.

   (a) Find an expression for a_0, $a_n + a_{n-1} + \ldots + a_0$, and $a_n - a_{n-1} + \ldots - a_1 + a_0$, in terms of the polynomial f.

   (b) Find an expression for $a_n + a_{n-1} + \ldots + a_1$ in terms of f.

   (c) Find an expression for a_1 in terms of f and its first derivative.

(d) Find an expression for a_j, $j \in \{2, 3, \ldots, n\}$ in terms of f and its higher derivatives.

8. A polynomial f of degree n and leading coefficient 1 has roots $1, 2, 3 \ldots n$. Find the constant term of the polynomial and the coefficient in front of the linear term. Is it possible to determine all the coefficients?

9. (*) Without using the angle formulae, prove that $\cos(3\theta) + 3\cos(\theta) = 4\cos^3(\theta)$ for any real θ. (Hint: de Moivre's theorem).

10. (*) Let $f(x) = c_3x^3 + c_2x^2 + c_1x + c_0$ be a polynomial with $c_3 > 0$ and $c_2, c_1, c_0 \in \mathbb{R}$. Assume $f(A) = f(B) = f(C) = 0$, where $A, B, C \in \mathbb{R}$ are distinct. Prove that

$$f'(A) + f'(B) + f'(C) > 0.$$