Extra Questions - MTH4104 Tutorial 2 - Relations

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The questions marked with a (*) are a little more tricky, but also more fun.

1. Which of the following are equivalence relations (i.e. satisfy reflexivity, symmetry and transitivity):

   (a) The set is $X = \{\text{Students of this term's Introduction to Algebra class}\}$, and we will say xRy if and only if the students x and y in X will get the same final mark for the course.

   (b) The relation on $\mathbb{R}$ that puts xRy if and only if $x - y = 0$.

   (c) The relation on $M(2, \mathbb{R})$, the set of two by two matrices with real entries, that has ARB if and only if $A = P^{-1}BP$ for some invertible matrix $P \in M(2, \mathbb{R})$.

   (d) Let $X = \{1, 2, \ldots 20\}$ and say xRy if and only if $x - y$ is divisible by 4.

   (e) $X = \{1, 2, \ldots 20\}$ and say xRy if and only if $5x - y$ is divisible by 4.

   (f) $X = \{\text{Polynomials with real coefficients}\}$, and fRg if and only if f divides g.

   (g) The relation on $\mathbb{Z}$ that puts xRy if and only if $|x - y|$ is a square number.

   (h) Let X be a non-empty set and $P(X)$ be the *power set* of X, i.e.the set of all subsets of X. Define a relation on $P(X)$ by xRy if and only if $x \cap y$ is non-empty. Is this an equivalence relation?

   (i) Let $X = \{\text{Mathematical Statements}\}$ and say aRb if and only if a implies b. Is this an equivalence relation? If not, tweak it a bit to make it one.

2. Consider the set $\mathbb{R}$ and the relation xRy if and only if $x - y$ is an integer. Which elements are related to the number 1.2? How about the number $\sqrt{2}$? Imagine that we treat every element that is related to 1.2 as the *same* element, and we choose the unique number in $[0, 1)$ that is related to 1.2 as a *representative* for all these elements. Which number would we choose? If we chose a representative for every $x \in \mathbb{R}$, what would the set of all representatives be?

3. Show that the relation on $\mathbb{Z}$ given by aRb if and only if $a+b$ is even is an equivalence relation.

4. How many equivalence relations on $\{1,2,3,4\}$ does there exist that contains $1R2$, $2R3$ and $3R4$?

5. Prove that the relation on X that satisfies xRy for all $x,y \in X$ is an equivalence relation.

6. (*) Let R be an equivalence relation on a set X. We define, for $x \in X$, the set $[x] := \{y \in X : xRy\}$, called the equivalence class of x.

   (a) Prove that $[x]$ is non-empty for every $x \in X$.

   (b) Prove that given any $x,y \in X$, either $[x]\cap[y] = [x]$ or $[x]\cap[y] = \varnothing$.

   (c) Let $X/R := \{[x] : x \in X\}$. Determine this set for the X and R as defined in problems 2, 3 and 5.

   (d) Define on $\mathbb{R}$ the relation xRy if and only if $\cos^2(x) + \sin^2(y) = 1$. Prove that this is an equivalence relation and find X/R.

7. (*) And to go back to polynomials, here is a fantastic question that I forgot to add in the last sheet. Let f be polynomial with coefficients that are positive integers. Prove that there exist two points x_1 and x_2 in $\mathbb{Z}^+$ for which knowing only the values of $f(x_1)$ and $f(x_2)$ determines the whole polynomial.