Extra Questions - MTH4104 Tutorial 3 - Equivalence Classes and Partitions

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The questions marked with a (*) are a little more tricky, but also more fun.

1. Fix a natural number n. Let R be the relation on $\mathbb{Z}$ which puts xRy if and only if $x - y = kn$ for some $k \in \mathbb{Z}$.
   (a) Prove that R is an equivalence relation.
   (b) Let $[x]$ denote the equivalence class of $x \in \mathbb{Z}$. Show that
   $$[x] = \{y \in \mathbb{Z} : \ y \text{ has the same remainder as } x \text{ when divided by } n\ \}.$$
   (c) Let $\mathbb{Z}/R = \{[x] : x \in \mathbb{Z}\}$. Show that this set is of size n, by showing that it can be identified with the set $\{0, 1, 2, \dots n - 1\}$. Into how many parts does this equivalence relation partition $\mathbb{Z}$?
   (d) (*) Define an addition operation $\dot{+}$ on $\mathbb{Z}/R$ by $[x]\dot{+}[y] := [x + y]$. Show this operation is well-defined; i.e. that it makes sense and does not lead to any contradictions.
   (e) (*) Come up with a well defined notion of multiplication in $\mathbb{Z}/R$. Prove that it is well-defined.
   (f) (*) Using your definition of multiplication, prove that when n is not prime, you could have two elements in $\mathbb{Z}/R$ which are not $[0]$ but whose product is $[0]$.
   (g) (*) Prove that this cannot happen when n is prime.
2. Let R be the relation on $\mathbb{Z}^2$ which puts $(a, b)R(c, d)$ if and only if $ad = bc$.
   (a) Prove that R is not an equivalence relation.
   (b) Is it an equivalence relation if defined on $\mathbb{Z} \times \mathbb{Z}\backslash\{0\}$ instead?
   (c) (*) Find $(\mathbb{Z} \times \mathbb{Z}\backslash\{0\})/R$, and identify this with a very well known set. How many partitions of $\mathbb{Z} \times \mathbb{Z}\backslash\{0\}$ does this relation create?
3. Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map, i.e. for all $x, y \in \mathbb{R}^d$ and $r \in \mathbb{R}$ we have that $f(x + y) = f(x) + f(y)$ and $f(rx) = rf(x)$.
   (a) Prove that $f(0) = 0$.
   (b) Define the *kernel* of f as the set $\ker(f) := \{x \in \mathbb{R}^d : f(x) = 0\}$, and the *image* of f as the set $\text{im}(f) := \{y \in \mathbb{R}^d : \exists x \in \mathbb{R}^d \text{ with } f(x) = y\}$.

Prove that the sum of any two elements in $\ker(f)$ remains in $\ker(f)$, and that multiplication by a real number of any element of $\ker(f)$ remains in $\ker(f)$.

(c) Prove the same result for $\mathrm{im}(f)$.

(d) Define a relation R on $\mathbb{R}^d$ by xRy if and only if $x - y \in \ker(f)$. Prove that this is an equivalence relation.

(e) (*) What is $\mathbb{R}^d/R$?

4. (*) How many equivalence relations are there on the set $\{1, 2, 3, 4\}$?

5. (*) A QMUL lecturer tells 2018 mathematicians that he will be making them stand in a line, ordered from the tallest mathematician to the shortest. The first mathematician is right at the back; she is the tallest and can see all the other 2017 mathematicians in front of her. The last mathematician is right at the front, he is the shortest and cannot see anyone. Every mathematician can see all the mathematicians in front of them, but not the ones behind.

   The QMUL lecturer tells the mathematicians that he will place a hat on everyone's head, and that it is either black or white. He will then ask each mathematician, starting at the back, what the colour of their hat is. If they get it right, they are set free. Whenever a mathematician answers, all the other mathematicians can hear the answer. Once the asking starts, no communication is allowed between the mathematicians, and only the words "black" or "white" can be uttered.

   However, the lecturer allows the mathematicians to discuss and come up with a strategy, before they are placed in line.

   (a) Find the strategy that will set free the maximum number of mathematicians. For example, one strategy is that every mathematician should say the colour of the hat of the person in front of him. But that will only save half of them. Can you do better?

   (b) Now assume that there are infinitely many mathematicians, and the rules are the same. Is there a strategy that can set free all but finitely many mathematicians? (Hint: equivalence classes).