# THE q-COMMUTING INVARIANTS OF SOME AUTOMORPHISMS OF FINITE ORDER ACTING ON THE q-DIVISION RING

ALI RAAD

Abstract. We consider the $q$-division ring in the two variables $x$ and $y$, $D_q(x,y)$. We focus on monomial automorphisms of this ring; represented by some element of $SL_2(\mathbb{Z})$, and examine further an order 5 automorphism that is not represented through $SL_2(\mathbb{Z})$. We thereafter create a new ring structure which allows us to simplify calculations in $D_q(x,y)$, and we use this in order to implement an algorithm which creates $q$-commuting invariants of our automorphisms outwith a non-constructive approach. We prove some new results.

## 1. Introduction

Consider a ring $R$, and an endomorphism $\alpha : R \to R$. Let $\delta$ be a left $\alpha$-derivation, ie. $\forall r, s \in R$, $\delta(rs) = \alpha(r)\delta(s) + \delta(r)s$. Consider the left Ore extension $R[x;\alpha,\delta] = \alpha(r)x + \delta(r)$. If we impose $\delta = 0$ and $\alpha(r) = qr$ for $q \neq 0$, $R = K[y]$, $K$ is a field of characteristic 0, we obtain the description of the *quantum plane* $K_q[x,y] = K[y][x;\alpha]$. We take the full field of fractions of it, $D_q(x,y) = Frac(K_q[x,y])$ and call this the *q-division ring*. This is the ring we will be studying.

Let $G$ be a group of finite order monomial automorphisms where $G \searrow D_q(x,y)$. The author in [1] focuses on automorphisms which can be represented as an element of $SL_2(\mathbb{Z})$, using the following definition:

**Definition 1** *A monomial automorphism $g$ acting on $D_q(x,y)$ is an element of $SL_2(\mathbb{Z})$ if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, and $gx = \hat{q}^{bd}y^b x^d$, $gy = \hat{q}^{ac}y^a x^c$ for $\hat{q}^2 = q$.*

The author in [1] presents four automorphisms of this type, given by

$$\tau : x \to x^{-1},\ y \to y^{-1}$$
$$\sigma : x \to y,\ y \to \hat{q}y^{-1}x^{-1}$$
$$\rho : x \to y^{-1},\ y \to x$$
$$\eta : x \to y^{-1},\ y \to \hat{q}yx$$

These automorphisms have orders 2, 3, 4, and 6 respectively. The author moves on to try and find invariant rings to these automorphisms.

**Definition 2** *Let $g$ be an automorphism of $D_q(x,y)$ of finite order. The invariant ring $D_q(x,y)^g = \{r \in D_q(x,y) : g(r) = r\}$.*

For each of the above automorphisms, the author manages to find two elements $u, v \in D_q(x,y)$ such that $\langle u, v\rangle = D_q(x,y)^g$ for $g$ one of the above automorphisms. Additionally, the author proves that $uv = qvu + \lambda$ for some $\lambda \in K^\times$ (the *quantum Weyl relation*, [1], Theorem 2.1), and thus manages to show that in fact

$$D_q(x,y)^g \cong D_q(x,y)$$

*Date*: September 20, 2017.

In ([1], Questions 4.12) the author defines an order 5 automorphism of $D_q(x,y)$ by

$$\pi : x \to y,\ y \to x^{-1}(y+q^{-1})$$

and conjectures that

$$D_q(x,y)^{\pi} \cong D_q(x,y)$$

This project focuses on two main issues. The first is to prove the above conjecture. But in order to do so, one realised that one needs a systematic way of creating $q$-commuting invariants of automorphisms of $D_q(x,y)$. Indeed some of the main results proved in [1] use computer algorithms found in [2] and are therefore sometimes highly unintuitive and highly non-constructive (see [1], Remark 3.5). Thus the second issue is to create a constructive method of finding $q$-commuting invariants of any given automorphism of finite order acting on $D_q(x,y)$.

## 2. The Order 5 Automorphism

Consider again the mapping

$$\pi : x \to y,\ y \to x^{-1}(y+q^{-1})$$

We have that $\pi(xy) = \pi(x)\pi(y) = y(x^{-1}(y+q^{-1})) = qx^{-1}y^2 + x^{-1}y$, and that $\pi(qyx) = q\pi(y)\pi(x) = q(x^{-1}(y+q^{-1}))y = qx^{-1}y^2 + x^{-1}y$, and thus $\pi$ is a well-defined homomorphism.

It is also order 5, since

$$\begin{aligned}
\pi^5(x) &= \pi^4(y) \\
&= \pi^3(x^{-1}(y+q^{-1})) \\
&= \pi^3(x^{-1}y + q^{-1}x^{-1}) \\
&= \pi^2(y^{-1}x^{-1}y + q^{-1}y^{-1}x^{-1} + q^{-1}y^{-1}) \\
&= \pi^2(q^{-1}x^{-1} + q^{-1}y^{-1}x^{-1} + q^{-1}y^{-1}) \\
&= \pi^2(q^{-1}(x^{-1} + y^{-1}x^{-1} + y^{-1})) \\
&= \pi(q^{-1}(y^{-1} + (y+q^{-1})^{-1}xy^{-1} + (y+q^{-1})^{-1}x)) \\
&= \pi(q^{-1}(y^{-1} + q^{-1}y^{-1}(y+q^{-1})^{-1}x + (y+q^{-1})^{-1}x)) \\
&= \pi(q^{-1}(y^{-1} + (q^{-1}y^{-1}+1)(y+q^{-1})^{-1}x)) \\
&= \pi(q^{-1}y^{-1} + (q^{-1}y^{-1}+1)(qy+1)^{-1}x) \\
&= \pi(q^{-1}y^{-1} + q^{-1}y^{-1}(1+qy)(qy+1)^{-1}x) \\
&= \pi(q^{-1}y^{-1}(1+x)) \\
&= q^{-1}(y+q^{-1})^{-1}x(1+y) \\
&= (qy+1)^{1}x(1+y) \\
&= (qy+1)^{-1}(1+qy)x \\
&= x
\end{aligned} \tag{1}$$

**Proposition 1** *Consider a mapping $f$ in $D_q(x,y)$ given by*

$$f : x \to y,\ y \to f_q(x,y)$$

*for some $f_q(x,y) \in D_q(x,y)$. Let $n \in \mathbb{N}$ be the minimal natural number satisfying $f^n(x) = x$. Then $f$ has order $n$.*

*Proof.* If $f^n(y) = y$ also then $f$ has order $n$. But clearly $f^n(y) = f^n(f(x)) = f(f^n(x)) = f(x) = y$. □

By Proposition 1, $\pi$ has order 5. Therefore $\pi$ is a bijection and so an automorphism on $D_q(x, y)$. It is conjectured in ([1], Questions 4.12) that

$$D_q(x, y)^\pi \cong D_q(x, y)$$

2.1. **Strategy to Prove the Conjecture.** What follows is the strategy adopted in order to prove the conjecture. We base this strategy on the proof in ([1], Theorem 2.1).

We want to find two invariants $\theta_1, \theta_2 \in D_q(x, y)$ (ie. $\pi(\theta_1) = \theta_1,\ \pi(\theta_2) = \theta_2$) such that if we declare $R = \langle \theta_1, \theta_2 \rangle$ then it will be that $R = D_q(x, y)^\pi$. In order for this to hold we first note that $R \subseteq D_q(x, y)^\pi \subset D_q(x, y)$ and so

$$[D_q(x, y) : R] = [D_q(x, y) : D_q(x, y)^\pi] \times [D_q(x, y)^\pi : R]$$

Hence if we show that $[D_q(x, y) : R] = 5$ that will imply that $[D_q(x, y)^\pi : R] = 1$ and so $R = D_q(x, y)^\pi$. We then want to define a quintic extension by

$$L := R[a; \gamma]/(a^5 + \lambda_1 a^4 + \lambda_2 a^3 + \lambda_3 a^2 + \lambda_4 a + \mu)$$

for an $a \notin R$, $a^5 \in R$ and $\gamma$ a certain automorphism on $R$. It is crucial that we find such an $a$ satisfying $R\langle a \rangle = \langle x, y \rangle$, because that would imply that $R\langle a \rangle = D_q(x, y)$ and thus $L$ will be of order 5 and so $[D_q(x, y) : R] = 5 \implies R = D_q(x, y)^\pi$, and so as a $k$-algebras we will have that

$$D_q(x, y)^\pi \cong D_q(x, y)$$

## 3. The Symmetrizing Idempotent

This section wil explore in depth the invariants $\theta_1, \theta_2 \in D_q(x, y)$ that we described in 2.1.

**Definition 3** *Let $g$ is an automorphism of $D_q(x, y)$ with order $n \in \mathbb{N}$. The symmetrizing idempotent is the function*

$$e_g := \sum_{i=0}^{n-1} g^i$$

*where $g^0 = id$.*

**Proposition 2** *Let $g$ be an automorphism of $D_q(x, y)$ with order $n \in \mathbb{N}$, then $\forall u \in D_q(x, y)$, $e_g(u) \in D_q(x, y)$ is invariant under $g$.*

*Proof.* Let $A := e_g(u) = \sum_{i=0}^{n-1} g^i(u)$. Now

$$g(A) = \sum_{i=0}^{n-1} g(g^i(u)) = \sum_{i=0}^{n-1} g^{i+1}(u) = \sum_{k=1}^{n-1} g^k(u) + g^n(u) = \sum_{k=0}^{n-1} g^k(u) = A$$

□

Proposition 2 tells us that we now have a well-defined method of creating invariants for automorphisms on $D_q(x, y)$. In light of the strategy outlined in 2.1 we note some fundamental restrictions:

(1) The element $u \in D_q(x,y)$ cannot be chosen arbitrarily. In fact if we want that $\theta_1 = e_g(u)$ and $\theta_2 = e_g(v)$ for some $u, v \in D_q(x,y)$ then we must have that $\theta_1\theta_2 = q\theta_2\theta_1$. This is a highly non-trivial restriction, even for a simple automorphism on $D_q(x,y)$ (see below).

(2) There is no guarantee that $R = \langle\theta_1, \theta_2\rangle = D_q(x,y)^g$. In fact we may need more invariants than just two, or we may need to manipulate other invariants of $g$ in order to find all the linearly independent invariants of $g$. This restriction will also be addressed in further detail below.

(3) According to the strategy in 2.1 we would also require an element $\zeta \in D_q(x,y)$ such that $R\langle\zeta\rangle = \langle x, y\rangle$. It is clear that the choice of $\zeta$ is dependant of the choice of $\theta_1$ and $\theta_2$.

Let us explore restriction 1 in some more depth. Consider again the automorphisms $\tau$ and $\sigma$ from section 1. If we use the symmetrizing idempotent on $x, y \in D_q(x,y)$ for $\tau$ and we use the idempotent on $x, x^{-1} \in D_q(x,y)$ for $\sigma$ we obtain

$$\theta_{\tau 1} := e_\tau(x) = x + \tau(x) = x + x^{-1}$$

$$\theta_{\tau 2} := e_\tau(y) = y + \tau(y) = y + y^{-1}$$

$$\theta_{\sigma 1} = e_\sigma(x) = x + \sigma(x) + \sigma^2(x) = x + y + \hat{q}y^{-1}x^{-1}$$

$$\theta_{\sigma 2} = e_\sigma(x^{-1}) = x^{-1} + \sigma(x^{-1}) + \sigma^2(x^{-1}) = x^{-1} + y^{-1} + \hat{q}yx$$

Notice that $\theta_{\tau 1}\theta_{\tau 2} \neq q\theta_{\tau 2}\theta_{\tau 1}$ and that $\theta_{\sigma 1}\theta_{\sigma 2} \neq q\theta_{\sigma 2}\theta_{\sigma 1}$. In fact the invariants that the author in [1] uses are highly non-intuitive and the result of long computation in Magma (see [2]). This illustrates the fact that even for a relatively simple order 2 automorphism it is very difficult to implement a structured algorithm for creating useful invariants.

However note that the sum of the powers of $x$ and $y$ appearing in each of $\theta_{\tau 1}, \theta_{\tau 2}, \theta_{\sigma 1}$ and $\theta_{\sigma 2}$ is 0.

**Proposition 3** *Let $g$ be a monomial automorphism of order $n \in \mathbb{N}$ acting on $D_q(x,y)$. If $\theta(x,y) \in D_q(x,y)$ is an invariant of $g$ and belongs to the image of $e_g$ then the sum of the powers of $x$ and $y$ together in $\theta(x,y)$ is 0.*

*Proof.* Since $g$ is a monomial automorphism we know from Definition 1 that $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$. In particular it is diagonalizable with $g \sim D = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$. Hence $e_g \sim \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)^2 + \dots \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)^{n-1} = \left(\begin{smallmatrix} \sum_{i=0}^{n-1} \lambda_1^i & 0 \\ 0 & \sum_{i=0}^{n-1} \lambda_2^i \end{smallmatrix}\right) = \left(\begin{smallmatrix} \frac{\lambda_1^n - 1}{\lambda_1 - 1} & 0 \\ 0 & \frac{\lambda_2^n - 1}{\lambda_2 - 1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ because $n > 1$ and $g^n = Id$. Now because $\theta(x,y)$ is in the image of $e_g$, we have that $e_g(u) = u + gu + g^2u + \dots g^{n-1}u = \theta(x,y)$ for some $u \in D_q(x,y)$. Note that each $g^j u$ transforms the powers of $x$ and $y$ in $u$ linearly by Definition 1, and so the total sum of the powers corresponds to the entries of the matrix $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and so the result holds. □

Perhaps the converse to Proposition 3 is more useful; if an element $\theta(x,y) \in D_q(x,y)$ does not admit that the powers of $x$ and $y$ sum to 0, then it could not have been an invariant of a monomial automorphism, created using the symmetrizing idempotent.

**Proposition 4** *Let $g, h$ be automorphisms of finite order on $D_q(x,y)$. Let $g = aha^{-1}$ for some automorphism $a$. If $z \in D_q(x,y)$ is $h$ invariant, then $a(z)$ is $g$ invariant.*

*Proof.* We have that $g(a(z)) = aha^{-1}(a(z)) = ah(z) = a(z)$ because $z$ is $h$ invariant. $\square$

Proposition 4 tells us that there is a way of mapping invariant elements to invariant elements. This could sometimes be helpful in order to use a more effective invariant element; in accordance with the strategy in 2.1. Thus we have found ways in which we can lessen the extent of restriction 1.

Let us now explore restriction 2. In order to do so we will make use of a theorem due to Emmy Noether presented as Theorem 1.5.1 in [3]. Note that the theorem was originally formulated for commutative algebras, but it is not difficult to extend the result to our non-commutative $q$-division ring.

**Theorem 1 (E. Noether)** *Assume that $K$ is a field of characteristic 0. Let $V$ be a vector space over $K$. For any representation $\rho : G \to GL(V)$ of a finite group $G$, the invariant algebra $K(V)^G$ is generated by the homogenous invariants functions of degree less than or equal to the order of $G$.*

Theorem 1 resolves restriction 2 because it tells us that we only have to consider a finite number of elements to use the symmetrizing idempotent on in order to find invariant generators to the entire invariant ring $D_q(x, y)^G$.

Let us return to the automorphism $\pi$. If we denote $\theta_1 = e_\pi(x), \theta_2 = e_\pi(y)$ then

$$\theta_1 = x + y + x^{-1}y + q^{-1}(2x^{-1} + 2y^{-1} + y^{-1}x^{-1}) + x(y + 1)^{-1}(1 + y^{-1})$$

$$\theta_2 = x^{-1}+y^{-1}+(y+q^{-1})^{-1}x+q(x^{-1}+y^{-1}x^{-1}+y^{-1})^{-1}+(q^{-1}y^{-1}+x(y+1)^{-1}(y^{-1}+1))^{-1}$$

If we were to assume that these are the invariants that we are looking for (as in the strategy in 2.1) then we would need an $a \in D_q(x, y)$ such that $\langle\theta_1, \theta_2\rangle\langle a\rangle = \langle x, y\rangle$. There is no direct method of finding such an $a$, especially since we do not know if $\theta_1, \theta_2$ are sufficient to prove the conjecture that

$$D_q(x, y)^\pi \cong D_q(x, y)$$

One could theoretically use Magma and the codes in [2] to try and find an algebraic manipulation of $\theta_1$ and $\theta_2$ in order to get the desired results. In fact for many of the monomial automorphisms, this is a strategy adopted in [1]. However we will take a different approach and attempt to answer the following question: is there a generic method of generating useful $q$-commuting invariants of automorphisms over $D_q(x, y)$? We will commence by attempting to answer this question for monomial automorphisms.

## 4. The Smash Product

Let $G = \{id, g, g^2, \ldots, g^{n-1}\}$ be a group of finite order monomial automorphisms of order $n \in \mathbb{N}$ over $D_q(x, y)$. Recall that $D_q(x, y)$ is a ring over a field $K$ of characteristic 0. Now consider a ring $(R, \cdot) = (K^n, \cdot)$ equipped with componentwise multiplication. Let $T$ be a tensor product defined by

$$T := \mathbb{C}[G] \otimes K^n$$

Let $e \in T$ be defined by

$$e := \sum_{g\in G} ge_g$$

for $e_g \in K^n$. What we are doing is defining a symmetrizing idempotent with a vector space structure; hence there is a basis, given by the elements $e_g$. Let

$$e : D_q(x, y) \to D_q(x, y) \otimes K^n$$

$$a \mapsto e(a) = \sum_{g \in G} g(a)e_g$$

Define the *smash product*:

$$S : D_q(x, y) \otimes K^n \times D_q(x, y) \otimes K^n \to D_q(x, y) \otimes K^n$$

$$S(e(a), e(b)) = e(a) * e(b) = e(ab)$$

or

$$S(\sum_{g \in G} g(a)e_g, \sum_{g \in G} g(b)e_g) = \sum_{g \in G} g(a)e_g * \sum_{g \in G} g(b)e_g = \sum_{g \in G} g(ab)e_g$$

Now the symmetrizing idempotent can be desribed by a mapping:

$$i : D_q(x, y) \otimes K^n \to D_q(x, y)$$

$$e(a) \mapsto i(e(a)) = \|e(a)\|_1$$

$$i(\sum_{g \in G} g(a)e_g) = \sum_{g \in G} g(a)$$

**Proposition 5** *The mapping*

$$e : D_q(x, y) \to D_q(x, y) \otimes k^n$$

$$a \mapsto e(a)$$

*defines a homomorphism under the smash product*

*Proof.* By definition of the smash product, $e(ab) = S(e(a), e(b)) = e(a)*e(b)\ \forall a, b \in D_q(x, y)$. Now note that if $ab = qba$ then $e(ab) = e(qba) = qe(ba) = qe(b) * e(a) = e(a) * e(b)$. □

**Theorem 2** *Let $G = \{id, g, g^2, \ldots, g^{n-1}\}$ be a group of automorphisms on $D_q(x, y)$ with $g^n = id$. Let $r \in D_q(x, y)$ be invariant under $g$. Let*

$$e = \sum_{g \in G} ge_g \in T$$

*Assume that $\exists s \in D_q(x, y)$ such that $rs = qsr$. Define*

$$A := \frac{i(r)}{n},\ B := i(s)$$

*where $i$ is the symmetrizing idempotent as defined by the smash product. Then $A, B \in D_q(x, y)$ are invariant under $g$, and more importantly,*

$$AB = qBA$$

*Proof.* Note that

$$e(rs) = \sum_{g\in G} g(rs)e_g \tag{2}$$
$$= \sum_{g\in G} g(r)g(s)e_g$$
$$= \sum_{g\in G} rg(s)e_g$$

as $r$ is invariant under $g$.

Hence,

$$i(rs) = \|e(rs)\|_1 = \sum_{g\in G} rg(s) = r\sum_{g\in G} g(s)$$

Note that $i(r) = \sum_{g\in G} g(r) = \sum_{g\in G} r = nr$, so that $r = \frac{i(r)}{n}$, a $g$-invariant.

Note also that $i(s) = \sum_{g\in G} g(s)$ is another $g$-invariant because it is created using the symmetrizing idempotent. Hence,

$$i(rs) = r\sum_{g\in G} g(s) = \frac{i(r)}{n}i(s) = AB$$

Now

$$e(qsr) = \sum_{g\in G} g(qsr)e_g = q\sum_{g\in G} g(s)re_g$$

$$\implies i(qsr) = \|e(qsr)\|_1 = q\sum_{g\in G} g(s)r = qi(s)\frac{i(r)}{n}$$

Since $rs = qsr$ by assumption we must have that

$$i(rs) = i(qsr)$$

$$\implies AB = qBA$$

where $A, B \in D_q(x, y)$ are $g$-invariant. □

Theorem 1 tells us that given only one $g$-invariant element $r$, we are able to explicitly construct an $A$ and a $B$ satisying $AB = qBA$, and both of them being $g$-invariant, as long as there exists an $s \in D_q(x, y)$ satisfying $rs = qsr$. Creating the $g$-invariant $r$ is a trivial matter, as $i(\omega)$ is invariant for any $\omega \in D_q(x, y)$. What is needed to complete the argument is a way of creating an $s$ satisfying $rs = qsr$ given an arbitrary $r \in D_q(x, y)$, which leads us to the next section.

## 5. An Algorithm which Creates q-commuting invariants

We start by assuming that $\exists F \in D_q(x, y)$ which we know explicitly:

$$F = \sum_{i\geq N} a_i x^i$$

with $a_N \neq 0$. Assume

$$a_{N+k} = \lambda_{N+k} y^{R_{N+k}},\ \lambda_N \neq 0,\ \lambda_{N+k} \in K,\ R_{N+k} \in \mathbb{Z}\ \ \forall k \in \mathbb{Z}_{\geq 0}$$

We want to find/create a $G \in D_q(y)((x))$ satisfying

$$FG = qGF$$

in accordance with Theorem 1. We will call such a $G$, if it exists, the *q-commuting partner of $F$*. We want

$$G = \sum_{j\geq M} b_j x^j,\ b_M \neq 0$$

Note that $G$ is a Laurent series, and the q-division ring is naturally embedded into the ring of Laurent series (see [1], 1). We have that

$$FG = qGF \iff (\sum_{i\geq N} a_i x^i)(\sum_{j\geq M} b_j x^j) = q(\sum_{j\geq M} b_j x^j)(\sum_{i\geq N} a_i x^i)$$

$$\implies \sum_{i\geq N, j\geq M} a_i \alpha^i(b_j) x^{i+j} = \sum_{i\geq N, j\geq M} q b_j \alpha^j(a_i) x^{i+j}$$

where $\alpha$ is the endomorphism explained in Section 1.

Consider that we solve the system by equating coefficients. Let us equate coefficients for the variable $x^{N+M+\epsilon}$ where $\epsilon \in \mathbb{Z}_{\geq 0}$. We obtain

$$\sum_{k=0}^{\epsilon} a_{N+k} \alpha^{N+k}(b_{M+\epsilon-k}) = \sum_{k=0}^{\epsilon} q b_{M+\epsilon-k} \alpha^{M+\epsilon-k}(a_{N+k})$$

which must hold $\forall \epsilon \in \mathbb{Z}_{\geq 0}$. Let us now insert the assumption that $a_{N+k} = \lambda_{N+k} y^{R_{N+k}}$ in order to obtain

$$\sum_{k=0}^{\epsilon} \lambda_{N+k} \alpha^{N+k}(b_{M+\epsilon-k}) y^{R_{N+k}} = \sum_{k=0}^{\epsilon} b_{M+\epsilon-k} \lambda_{N+k} q^{(M+\epsilon-k)R_{N+k}+1} y^{R_{N+k}}$$

This is now a polynomial equation in the variable $y$. We assume further that

$$y^{R_{N+k_1}} = y^{R_{N+k_2}} \iff k_1 = k_2$$

so that we can solve the above equation by equating coefficients.

Fix an $\epsilon = p \in \mathbb{Z}_{\geq 0}$ and a $k = \bar{k} \in \{0, 1, \ldots, p\}$ where $\lambda_{N+\bar{k}} \neq 0$. We have that

$$\alpha^{N+\bar{k}}(b_{M+p-\bar{k}}) = b_{M+p-\bar{k}} q^{(M+p-\bar{k})R_{N+\bar{k}}+1}$$

$$\implies b_{M+p-\bar{k}} = \psi_{M+p-\bar{k}} y^{S_{M+p-\bar{k}}},\ \psi_{M+p-\bar{k}} \in K,\ S_{M+p-\bar{k}} \in \mathbb{Z}$$

If we assume further that $\psi_{M+p-\bar{k}} \neq 0$ for our specific choice of $p$ and $\bar{k}$ we are therefore able to obtain the equation

$$\psi_{M+p-\bar{k}} q^{(S_{M+p-\bar{k}})(N+\bar{k})} y^{S_{M+p-\bar{k}}} = \psi_{M+p-\bar{k}} y^{S_{M+p-\bar{k}}} q^{(M+p-\bar{k})R_{N+\bar{k}}+1}$$

$$\implies q^{(S_{M+p-\bar{k}})(N+\bar{k})} = q^{(M+p-\bar{k})R_{N+\bar{k}}+1}$$

$$\implies S_{M+p-\bar{k}}(N+\bar{k}) - (M+p-\bar{k})R_{N+\bar{k}} = 1$$

**Theorem 3** *For a given $F$ as defined above, a q-commuting parnter $G \in D_q(y)((x))$ will always have a monomial in $y$ as its $x$ coefficients. Furthermore, in order for such an $F$ to admit a q-commuting partner $G$, it must be that $gcd(N+\bar{k},\ R_{N+\bar{k}}) = 1$ in $F$, and $gcd(S_{M+p-\bar{k}},\ M+p-\bar{k}) = 1$ in the $G$ created.*

*Proof.* In the way we have defined $F$ (see above) and due to the restrictions we have placed on it, we saw from the above argument that in order for $G$ to exist, its coefficients must have the form $b_{M+p-\bar{k}} = \psi_{M+p-\bar{k}} y^{S_{M+p-\bar{k}}}$. From the derived equation

$$S_{M+p-\bar{k}}(N+\bar{k}) - (M+p-\bar{k})R_{N+\bar{k}} = 1$$

it is clear that $gcd(N+\bar{k},\ R_{N+\bar{k}}) = 1$ in $F$, and $gcd(S_{M+p-\bar{k}},\ M+p-\bar{k}) = 1$ in the $G$ created. □

**Theorem 4** *Let $F$ be an element of $D_q(x,y)$ as defined above. Then*

(1) *if $F$ is a monomial in $x$ and $y$ with $gcd(N, R_N) = 1$, a q-commuting partner $G$ always exists, and is monomial in $x$ and $y$. In particular $G = \psi_M y^{S_M} x^M$ with $NS_M - MR_N = 1$.*

(2) *if $F$ has at least 2 terms (i.e. $F = a_N x^N + a_{N+\bar{k}} x^{N+\bar{k}} + \ldots$), where $a_{N+\bar{k}} x^{N+\bar{k}} = \lambda_{N+\bar{k}} y^{R_{N+\bar{k}}} x^{N+\bar{k}}$, then a necessary condition for the existence of a q-commuting partner $G$ is that:*

   (a) *if $N = 0$, then $R_N = \pm 1$. If $R_N = 1$, then $M = -1$ and $S_M = \frac{1-R_{N+\bar{k}}}{\bar{k}}$ must be an integer. If $R_N = -1$, then $M = 1$ and $S_M = \frac{1+R_{N+\bar{k}}}{\bar{k}}$ must be an integer.*

   (b) *If $N \neq 0$ then $M = \frac{-\bar{k}}{\bar{k}R_N + N(R_N - R_{N+\bar{k}})}$, $S_M = \frac{R_N - R_{N+\bar{k}}}{\bar{k}R_N + N(R_N - R_{N+\bar{k}})}$ must both be integers.*

*Proof.* We have that the equation

$$S_{M+p-\bar{k}}(N+\bar{k}) - (M+p-\bar{k})R_{N+\bar{k}} = 1$$

is valid when $\lambda_{N+\bar{k}},\ \psi_{M+p-\bar{k}} \neq 0$ for our choice of $p$ and $\bar{k}$. If $F$ is a monomial in $x$ and $y$, then it must be that $\bar{k} = p = 0$, and so we must solve the Diophantine equation

$$S_M N - MR_N = 1$$

which is guaranteed to have integral solutions for $S_M$ and $M$ because $gcd(N, R_N) = 1$.

Now assume $F$ has at least 2 terms; fix a $\bar{k} \in \mathbb{Z}^+$ satisfying $\lambda_{N+\bar{k}} \neq 0$ and *choose* $p = \bar{k}$. This is a valid choice as then $\psi_{M+p-\bar{k}} = \psi_M \neq 0$, and so the equation

$$S_M(N+\bar{k}) - MR_{N+\bar{k}} = 1 \quad \ldots (3)$$

must hold. But we know that the equation

$$S_M N - MR_N = 1 \quad \ldots (4)$$

must also hold because we can still choose $p = \bar{k} = 0$. Subtracting (4) from (3) we obtain

$$S_M \bar{k} - M(R_{N+\bar{k}} - R_N) = 0 \quad \ldots (5)$$

where $\bar{k}, R_{N+\bar{k}}, R_N \in \mathbb{Z}$ are assumed to be known beforehand.

If $N = 0$ then (4) $\implies -MR_N = 1$ and so $R_N = \pm 1 \implies M = \mp 1$. The case $R_N = 1,\ M = -1$ implies in (5) that

$$S_M \bar{k} + R_{N+\bar{k}} - 1 = 0$$

$$\implies S_M = \frac{1 - R_{N+\bar{k}}}{\bar{k}}$$

which should be an integer if a q-commuting partner to $F$ should exist. In the case $R_N = -1,\ M = 1$, (3) implies that

$$S_M\bar{k} - R_{N+\bar{k}} - 1 = 0$$
$$\implies S_M = \frac{1 + R_{N+\bar{k}}}{\bar{k}}$$

which should be an integer.

If $N \neq 0$ then (4) implies that

$$S_M = \frac{1 + MR_N}{N} \quad \ldots (6)$$

and substituting this into (5) we obtain

$$\bar{k}(\frac{1 + MR_N}{N}) = M(R_{N+\bar{k}} - R_N)$$
$$\implies \frac{\bar{k}}{N} + M(\frac{\bar{k}R_N}{N}) = M(\frac{NR_{N+\bar{k}} - NR_N}{N})$$
$$\implies M = \frac{-\bar{k}}{\bar{k}R_N + N(R_N - R_{N+\bar{k}})}$$

which must be an integer if $G$ is to exist. Substituting this result into (6) we obtain

$$S_M = \frac{1 + (\frac{-\bar{k}}{\bar{k}R_N + N(R_N - R_{N+\bar{k}})})R_N}{N}$$

which can be simplified into

$$S_M = \frac{R_N - R_{N+\bar{k}}}{\bar{k}R_N + N(R_N - R_{N+\bar{k}})}$$

which must be an integer in order for $G$ to exist. This finishes the proof. □

## 6. Some Concluding Remarks

The aim of the summer project was to prove the conjecture that

$$D_q(x,y)^\pi \cong D_q(x,y)$$

In order to do so we outlined a strategy for a proof, based on ([1], Theorem 2.1). This strategy required the creation of two q-commuting invariants. Computer algorithms seemed unlikely to yield the required invariants so one explored a new method of creating such invariants. Under the definition of a new ring (in the form of a tensor product) one proved that in theory one was able to algorithmically create useful q-commuting invariants. In order for the theory to work in practice, one needed an algorithm to create a q-commuting partner to any $F \in D_q(x,y)$ which satisfied some chosen properties. One proved that such a partner exists if some fundamental conditions were satisfied. Due to lack of time, the project could not continue beyond Theorem 4. However we leave the reader with a few open questions:

**Questions**

- Will the $G$ created using Theorem 4 be rational, and hence will it be an element of $D_q(x,y)$?
- Can one drop the assumption that $y^{R_{N+k_1}} = y^{R_{N+k_2}} \iff k_1 = k_2$ which was used to prove Theorem 3 and still manage to find a way of creating a q-commuting partner to $F$?

- After the $G$ has been created using Theorem 4, and after one uses Theorem 2 to create elements $r,\ s \in D_q(x,y)$ satisfying $rs = qsr$, can one find an effective way to use $r$ and $s$ for the strategy outlined in 2.1? Hence can one prove that $D_q(x,y)^\pi \cong D_q(x,y)$?

## References


[1] `http://arxiv.org/pdf/1310.5071v2.pdf`

[2] `http://www.maths.manchester.ac.uk/~sfryer/magma_writeup.pdf`

[3] `http://math.univ-bpclermont.fr/~fdumas/fichiers/CIMPA.pdf`