Extra Questions - MTH4104 Tutorials 4 and 5 - Divisibility

Ali Raad
e-mail: a.i.raad@qmul.ac.uk

Queen Mary University of London

February 7, 2018

The questions marked with a (*) are a little more tricky, but also more fun.

1. Prove that the product of any 3 consecutive integers is divisible by 6.
2. Let $\mathbb{Z}_4[x]$ denote the set of polynomials in x with coefficients in $\mathbb{Z}_4$.
   (a) Prove that there is a non-zero polynomial in this set which multiplies $2x^2 + 2$ to get 0.
   (b) Prove that there is no non-zero polynomial in $\mathbb{Z}_4[x]$ which multiplies the polynomial $2x + 1$ to get 0.
   (c) We will call the space $\mathbb{Z}_n$ for $n \in \mathbb{N}$ *good* if there are no non-zero elements a, b in the space for which $ab = 0$. For which $n \in \mathbb{N}$ do we get good spaces? Prove your result.
3. (*) There are 2018 warriors who were captured in battle. Refusing to surrender, they decide to eliminate themselves in a particular manner. They sit around a circular table, in a clockwise manner starting with Warrior 1 and ending with Warrior 2018. Then Warrior 1 eliminates Warrior 2, Warrior 3 eliminates Warrior 4 etc, each person eliminating the person to their left. This keeps going until one Warrior is left. Which number does this final Warrior have?
4. Let a and b be positive integers where a is even and such that $\gcd(2a, 2b) = 70$. Find $\gcd(a, 2b)$.
5. Let a and b be positive integers. Prove that $\gcd(\frac{a}{\gcd(a,b)}, \frac{b}{\gcd(\text{a,b})}) = 1$.
6. (a) Prove that a positive integer z is divisible by 3 if and only if the sum of its digits are divisible by 3.
   (b) Prove that a positive integer z is divisible by 9 if and only if the sum of its digits are divisible by 9.
   (c) Find a condition for when a positive integer is divisible by 11, and prove it.
7. Let a and b be positive integers, such that $\gcd(a, 30) = 1$. Prove that $a^4 + 59$ is divisible by 60.
8. (*) Find the smallest integer $n \geq 3$ for which the following property holds: there exist two distinct elements in $\{1, 2, \ldots n\}$ whose product equals the sum of all the remaining $n - 2$ elements.

9. Prove that there are no square numbers of the form $4k + 3$ or $4k + 2$ where $k \in \mathbb{N}$.

10. (*) Find the sum of all positive integers m for which $m^2 + 2m + 2$ divides $m^3 + 4m^2 + 4m - 14$.

11. Let x_m and y_m be integers defined by $x_m + y_m\sqrt{2} = (1 + \sqrt{2})^m$. Prove that $\gcd(x_m, y_m) = 1$.

12. (*) Let a and b be positive integers. Prove that $\frac{\gcd(a,b)}{b}\binom{b}{a}$ is always an integer.