Extra Questions - MTH4104 - Elementary Rings and Fields

Ali Raad
e-mail: a.i.raad@qmul.ac.uk

Queen Mary University of London

February 27, 2018

The questions marked with a (*) are a little more tricky, but also more fun.

1. Is there a ring with one element? What must this element be? (Such a ring is called a *trivial ring.*)
2. Using the definition of a field given in your notes, prove that there is no field with one element. Which law can we remove in order to allow a field with one element?
3. Is $\mathbb{N}$, with usual addition and product, a ring?
4. Let R be a ring. Is $\mathrm{Mat}_n(R)$, the set of all n by n matrices with entries in R, with usual matrix multiplication and addition, a ring? Give an example of an n and an R that makes $\mathrm{Mat}_n(R)$ into a field.
5. Define an addition and multiplication operation on $R_1 \times R_2$, where R_1 and R_2 are rings, that makes $R_1 \times R_2$ a ring. If R_1 and R_2 are fields, then is $R_1 \times R_2$ also a field?
6. Let R be a ring such that $r^2 = r$ for all $r \in R$. Prove that R is commutative (i.e. that $rs = sr$ for all $r, s \in R$).
7. (*) Let S be the set whose elements are all possible subsets of $\{1, 2, \ldots, 10\}$, including the empty subset. Find a multiplication and addition operation on S that will turn it into a ring.
8. (*) Let R be a ring, but not a field. Assume that for all non-invertible elements r in R we have the condition that $r = r^2$. Prove that for all invertible $i \in R$ and all non-zero, non-invertible $r \in R$ we have that $i + r$ is not invertible. Use this to prove that R is commutative.