Extra Questions - MTH4104 - Elementary Group Theory

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The questions marked with a (*) are a little more tricky, but also more fun.

1. Let G and H be groups. Find an operation on $G \times H$ that turns it into a group.

2. Let $SL_2(\mathbb{R})$ denote the set of 2 by 2 real matrices with determinant 1. Show that this set, with the multiplication operation, is a group.

3. Let R be a ring with identity. Show that the invertible elements of R form a group.

4. Let G be a group. We say that N is a *normal* subgroup of G if it is a subgroup and for all $n \in N$ and $g \in G$ we have that $g^{-1}ng \in N$. For an element $g \in G$ and a subgroup $H \leq G$, define $gH := \{gh \mid h \in H\}$, and $Hg := \{hg \mid h \in H\}$.

   (a) Show that the following are equivalent:

   - N is normal
   - For all $g, h \in G$, $gh \in N \iff hg \in N$.
   - For all $g \in G$ we have $g^{-1}Ng \subseteq N$.
   - For all $g \in G$ we have $g^{-1}Ng = N$.
   - For all $g \in G$ we have $Ng = gN$.

   (b) The *centre* of a group G, $Z(G)$, is the set of those elements of G that commute with every element of G. Show that $Z(G)$ is non-empty and that it is a normal subgroup of G.

   (c) Give a type of group for which every subgroup is normal.

   (d) If H and K are subgroups of G, is $HK := \{hk \mid h \in H, k \in K\}$ also? If not, find a condition on H and/or K that makes HK into a subgroup. Find a condition that makes HK into a normal subgroup.

5. Prove that if G is an infinite group then it has infinitely many subgroups.

6. (*) If G is a finite group and H is a normal subgroup of G and K is a normal subgroup of H, is it always true that K is a normal subgroup of G?

7. Let G be a finite group where 40 of its elements have order 5. How many different subgroups of order 5 does G have?

8. Prove that if $g^2h^2 = (gh)^2$ for every g and h of a group G, then G is abelian.

9. (*) Define a *groupoid* as a set G which contains a map

$$\star : G \to G; \;\; g \to g^\star$$

and a certain subset $A \subset G \times G$, with a map

$$p : A \to G, \;\; (g,h) \to gh$$

satisfying the following conditions:

- $(g^\star)^\star = g$ for all $g \in G$
- $(g, g^\star), (g^\star, g) \in A$ for all $g \in G$
- If $(g,h) \in A$ and $(h,k) \in A$ then both (gh,k) and (g,hk) belong to A and $p((gh,k)) = p((g,hk))$
- If $(g,h) \in A$ then $p((g^\star g,h)) = h$ and $p((g,hh^\star)) = g$

Define $G^0 := \{gg^\star : g \in G\} \subset G$, and $s(g) := g^\star g$, $r(g) = gg^\star$.

(a) Prove that $(g,h) \in A \iff s(g) = r(h)$.

(b) Prove that for $(g,h) \in A$, we have that $(h^\star, g^\star) \in A$ and $(gh)^\star = h^\star g^\star$.

(c) Prove that if G is a group, then it is also a groupoid with $G^0 = \{e\}$. Prove further that if G is a groupoid with $G^0 = \{e\}$ then in fact it is a group.

10. (*) Consider an equilateral triangle with vertices 1, 2 and 3, sitting in $\mathbb{R}^2$. Define the operation r on the triangle to be rotation by 120 degrees. Define the operation s to be reflection along the perpendicular bisector of $\overline{12}$. Let id be the operator that does nothing to the triangle.

Define $S_3 := \{id, r, r^2, s, s \circ r, s \circ r^2\}$. Prove that S_3 is a group, where $\circ$ denotes composition. Find an injective group homomorphism $S_3 \to \mathrm{Mat}_3(\mathbb{R})$.

Now imagine that the vertices were cities, where we will draw an arrow from i to j ($i, j \in \{1, 2, 3\}, i \neq j$), to mean there is a road from i to j. (Note that if there is a road i to j that doesn't necessarily mean a road exists from j to i). In total, there are 64 such types of triangles with arrows. Given such a triangle t, let T be its *adjacency* matrix, ie the 3 by 3 matrix with a 1 in the (i, j) position if there is a connection i to j, and 0 otherwise.

Note that if an element of S_3 acts on a triangle t, we will get a new adjacency matrix for the new triangle. Find a way to act on the original adjacency matrix, T, by elements of the image of the homomorphism you found, to respect such transformations.